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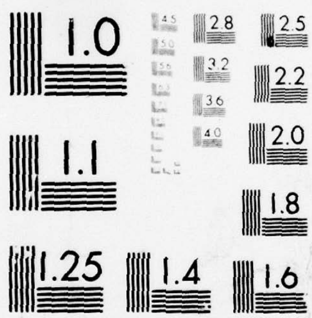

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10 by Emanuel Parzen

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Sponsored by the Office of Naval Research

Professor Emanuel Parzen, Principal Investigator

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Time Series Analysis Methods and Applications: Bibliography of Books in English

by

Emanuel Parzen

Introduction

This bibliography, together with a recent report by Parzen entitled "Time Series Analysis Methods and Applications: Increased Interdisciplinary Interaction Could Stimulate Research Breakthroughs", have been prepared to help planning for future directions of research and education in statistical time series analysis methods and applications.

This bibliography provides a list of books available in English, and reproduces from each book its title page and table of contents. It also includes prefaces and references from many books.

This bibliography should not be regarded as an exhaustive list. To expedite its preparation, the current list includes only books which were easily available at this time. Since it is hoped to prepare a supplementary list, suggestions for additions to the list are welcome.

For their invaluable assistance, I would like to thank my secretary, Nan Mitchell, and the students enrolled in my Fall 1979 time series course.

1938 - Wold, Herman
A Study in the Analysis of Stationary Time Series.

1941 - Davis, Harold T.
The Analysis of Economic Time Series.

1949 - Wiener, Norbert
Extrapolation, Interpolation, and Smoothing of Stationary Time Series

1957 - Grenander, Ulf, and Rosenblatt, Murray
Statistical Analysis of Stationary Time Series.

1957 - Quenouille, M.H.
The Analysis of Multiple Time-Series.

1958 - Blackman, R.B., and Tukey, J.W.
The Measurement of Power Spectra

1960 - Hannan, E.J.
Time Series Analysis

1962 - Yaglom, A.M.
An Introduction to the Theory of Stationary Random Functions.

1963 - Brown, Robert Goodell
Smoothing, Forecasting and Prediction of Discrete Time Series.

1963 - Whittle, P.
Prediction and Regulation by Linear Least-Square Methods.

1964 - Granger, C.W.F., and Hatanaka, M.
Spectral Analysis of Economic Time Series.

1964 - Gregg, J.V., Hossell, C.H., and Richardson, J.T.
Mathematical Trend Curves: an Aid to Forecasting.

1966 - Simpson, Stephen Milton Jr.
Time-Series Computations in FORTRAN and FAP.

1967 - Albert, Arthur E., and Gardner, Leland A. Jr.
Stochastic Approximation and Nonlinear Regression.

1967 - Harris, Bernard
Spectral Analysis of Time Series.

1967 - Ivakhnenko, A.G., and Lapa, V.G.
Cybernetics and Forecasting Techniques.

1967 - Merriam, Daniel F.
Computer Applications in the Earth Sciences: Colloquium on Time-Series Analysis.

1967 - Parzen, Emanuel
Time Series Analysis Papers.

1967 - Robinson, Enders A.
Statistical Communication and Detection.

1967 - Rozanov, Yu.A.
Stationary Random Processes.

1968 - Jenkins, Gwilym M., and Watts, Donald G.
Spectral Analysis and its applications.

1968 - Robinson, Enders A.
Multichannel Time Series Analysis with Digital Computer Programs.

1969 - Fishman, George S.
Spectral Methods in Econometrics

1970 - Astrom, Karl J.
Introduction to Stochastic Control Theory.

1970 - Box, George E.P., and Jenkins, Gwilym M.
Time Series Analysis Forecasting and Control.

1970 - Brillinger, David R.
Time Series Data Analysis and Theory.

1970 - Hannan, E.J.
Multiple Time Series.

1971 - Anderson, T.W.
The Statistical Analysis of Time Series.

1971 - Bendat, Julius S., and Piersol, Allan G.
Random Data: Analysis and Measurement Procedures.

1973 - Kendall, Sir Maurice
Time-Series.

1973 - Nelson, Charles R.
Applied Time Series Analysis for Managerial Forecasting.

1974 - Chatfield, C.
The Analysis of Time Series: Theory and Practice.

1974 - Koopmans, L.H.
The Spectral Analysis of Time Series.

1975 - Kanasewich, E.R.
Time Sequence Analysis in Geophysics.

1976 - Bloomfield, Peter
Fourier Analysis of Time Series: An Introduction.

1976 - Claerbout, Jon F.
Fundamentals of Geophysical Data Processing.

1976 - Fuller, Wayne A.
Introduction to Statistical Time Series.

1976 - Gilchrist, Warren
Statistical Forecasting.

1976 - Kashyap, R.L., and Rao, A. Ramachandra
Dynamic Stochastic Models from Empirical Data.

1976 - Mehra, Raman K., and Lainiotis, Dimitri G.
System Identification.

1976 - Montgomery, Douglas C., and Johnson, Lynwood A.
Forecasting and Time Series Analysis.

1976 - Pindyck, Robert S., and Rubinfeld, Daniel L.
Econometric Models and Economic Forecasts.

1977 - Granger, C.W.J., and Newbold, Paul
Forecasting Economic Time Series.

1977 - Kailath, Thomas
Linear Least-Squares Estimation.

1978 - Childers, Donald G.
Modern Spectrum Analysis.

1978 - Findley, David F.
Applied Time Series Analysis.

1978 - Ostrom, Charles W. Jr.
Time Series Analysis: Regression Techniques.

1978 - Otnes, Robert K., and Enochson, Loren
Applied Time Series Analysis.

1979 - Robinson, Enders A., and Silvia, Manuel T.
Digital Foundations of Time Series Analysis.

1979 - Silvia, Manuel T., and Robinson, Enders A.
Deconvolution of Geophysical Time Series in the Exploration for Oil and Natural Gas.

1979 - Nerlove, Marc, Grether, David M., and Carvalho, José L.
Analysis of Economic Time Series.

1938

A STUDY IN THE ANALYSIS OF STATIONARY TIME SERIES

BY

HERMAN WOLD

SECOND EDITION

WITH AN APPENDIX BY

PETER WHITTLE

ALMQVIST & WIKSELL

STOCKHOLM

PREFACE.

In a sequence of fundamental memoirs, G. Udny Yule, the eminent English statistician, has proposed certain methods of time series analysis which are of an essentially wider scope than the classical methods used in the search for periodicities. The basis of the new methods is a concept of flexible periodicity which in an ideal case reduces to the classical, functionally rigid periodicity. The importance and the broad applicability of the new ideas has been stressed particularly in subsequent discussion of the nature of business cycles.

In the recent rapid development of the theory of probability, the production of A. Khintchine and A. Kolmogoroff represents a genuine discontinuity. A firm, axiomatic foundation has been obtained for the theory; other important results belong to the theory of random processes, i. e. hypothetical models for the analysis of time series. In accordance with the great diversity of time series, the main types of random process are of quite different structure.

In the theory of probability, the approaches of G. U. Yule fall under the heading of the stationary random process as defined and studied by A. Khintchine. The present work might be described as a trial to subject the fertile methods of empirical analysis proposed by Yule to an examination and a development by the use of the mathematically strict tools supplied by the modern theory of probability. This statement, however, implies no valuation of the results and should be regarded rather as a tribute to my sources of inspiration and to the traditions of my milieu of study.

My most sincere thanks are due to my teacher, Professor Harald Cramér. His brilliant courses, distinguished by a spirit of realism combined with penetrating logic, have laid the foundation for my further work. As far as the present thesis is concerned, this is true not only in general but also in respect to particular parts thereof, as indicated by the references to his 1933 course on Time Series Analysis. I wish to evidence my deep gratitude to Professor Cramér also for the encouragement and interest shown me at all

times, and culminating in his detailed reading of the first version of the manuscript. Our subsequent discussions have caused a revision particularly of the treatment of questions of convergence in probability.

To the Royal Swedish Academy of Sciences I want to express my respectful gratitude for a generous grant covering a substantial part of the expenses for printing and numerical calculation.

I am greatly indebted to my friends and colleagues Mr. G. ELFVING, Mr. W. FELLER and Mr. O. LUNDBERG for numerous stimulating discussions and for having read the manuscript and corrected many errors. I have also profited to a great extent by consultations with a large number of research workers in the different fields touched upon in the thesis. These obligations are, however, too comprehensive and indefinite to be expressed in detail.

Stockholm, July 1938.

H. W.

PREFACE TO THE SECOND EDITION.

Stationary processes having in the last 15 years been the subject of intensive research, important results have been obtained both regarding their theory and their many fruitful applications. In presenting a new edition of my thesis, the recent development is briefly dealt with in Appendices 1–2 (these replace Appendices A–B of the first edition, which were devoted to special topics), whereas the main text is left unaltered, except for a slight revision that makes use only of material available in 1938. I am greatly indebted to Dr. WHITTLE for writing Appendix 2, in which two main lines of progress are surveyed, viz. spectral theory and methods of statistical inference. The short Appendix 1 comments, by way of numbered foot-notes, a few specific points in the main text.

The first edition had the dual form of an expository survey and a research report. It is hoped that the second edition may still serve as an introduction to the theory and the applications of stationary processes.

Uppsala, March 1953.

H. W.

Contents.

List of Theorems.

List of Graphs.

1941

THE ANALYSIS OF ECONOMIC TIME SERIES

By

HAROLD T. DAVIS

THE PRINCIPIA PRESS, INC.
BLOOMINGTON, INDIANA
1941

PREFACE

The object of the present volume is to set forth in some detail the present status of the problem of analyzing and interpreting that very extensive set of data known as economic time series. This perplexing problem has engaged the attention of economists and statisticians for many years, but the extraordinary intensity with which it has been attacked during the past decade attests the importance which it has for modern economic development.

Since its beginning the laboratory of the Cowles Commission for Research in Economics has had as a major interest the investigation of the nature and action of stock price series. In the course of this investigation a number of interesting but difficult problems were encountered concerning the nature of economic time series in general, and the relation of these series to the basic postulates of economic theory in particular. To most of these questions only partial answers were discovered in the literature and in many cases these answers were not accompanied by careful statistical analyses. Therefore, it seemed to the author that a systematic treatise on the nature of economic series might fill a present need.

To one who works with statistical data it soon becomes apparent that the conclusions derived at the end of a process of analysis are intimately related to the postulates which underlie the tools employed in the investigation. The employment of a linear trend for the reference of residuals, or the graduation of a series of production data by means of the logistic curve, implies economic assumptions which must be carefully defined and subjected to realistic criticism. That is to say, conclusions mathematically derived are no better than the postulates upon which they rest. Hence it has seemed necessary to make a careful re-examination of the various mathematical devices which have been used in the study of economic data in order to appraise their weakness and their strength, and to define the range of their validity.

There is a perpetual fascination in economic time series, derived not only from their immense importance in the lives of all of us, but also from their statistical nature. Differing from the series encountered in the experiments of physical science, every economic time series possesses a large random element. But the series themselves are not random, in spite of some popular belief to the contrary, nor are

they sufficiently regular to satisfy most mathematical postulates. Hence, in many instances, the analysis must proceed from a description of the differences between random series and series that are not random. Correlations take the place of functions and serial relationships replace the more familiar functional equations of the exact sciences.

In the course of preparing so extensive a manuscript the author has become indebted to many people. Foremost among these is Mr. Alfred Cowles, president of the Cowles Commission, who for nearly a decade has liberally supported a scientific laboratory devoted to the investigation of problems in economic theory and economic statistics. His personal interest in these investigations and his own scientific contributions to the subject have been a source of inspiration and satisfaction to the author.

From Mr. Dickson H. Leavens, managing editor of *Econometrica* and research associate of the Cowles Commission, the author has received services too numerous to mention. Mr. Leavens assumed full editorial supervision of the manuscript and the planning of the charts is to be credited entirely to him.

During the preparation of the book the author received many suggestions from Dr. C. F. Roos, former research director of the Cowles Commission, and from Professor T. O. Yntema, the present research director. Their broad knowledge of economic problems was placed generously at his disposal.

A special debt of thanks is also due Professor Gerhard Tintner of Iowa State College, who read the entire proof carefully and offered many valuable suggestions. His exceptionally wide acquaintance with economic and statistical literature, especially that of European schools, has made his criticism of great value.

To Mr. Herbert E. Jones, research associate of the Cowles Commission, the author is indebted for a number of essential contributions to the book. Mr. Jones undertook a thorough investigation of problems relating to the theory and application of serial correlation. In particular, he studied the properties of random series and then applied his analysis to the problem of determining the nature of the structural elements in economic time series. Much of the material in Chapters 3 and 4 is derived from his studies.

Throughout the long and arduous calculations presented at many places in the book the laboratory staff of the Cowles Commission has played an indispensable role. The brunt of this work has been assumed by Mr. Forrest Danson, research associate of the Cowles Commission and director of the computing laboratory. The author is especially

indebted to him. In this phase of the work numerous computations were made by Miss Emma Manning, Miss Anne M. Lescisin, Mr. Edward Morris, and Mrs. Martha Belschner Swanson. Miss Kathryn Withers did the arduous work of inking and lettering the charts and Miss Mary Jo Lawley helped in preparing the manuscript for the printer.

To the great experience of Professor Irving Fisher in monetary theory and to the statistical studies of Mr. Carl Snyder on economic trends and the theory of prices the author owes a special debt. From conversations with Professor Ragnar Frisch of Oslo, Norway, and from his writings, more particularly his studies of harmonic analysis, confluence analysis, and the dynamics of cycles, the author has derived many valuable suggestions. Professor J. W. Angell of Columbia University very kindly supplied the author with monetary data which would otherwise have been inaccessible to him.

The author would also like to acknowledge his appreciation of the critical advice received from Dr. John Smith, research associate of the Cowles Commission, who has brought to bear upon the analysis a broad knowledge of statistical sampling. His criticism has been particularly valuable in connection with some of the material in Chapter 5. From other colleagues in the research staff of the Cowles Commission many helpful suggestions have been received. Professor Francis McIntyre, Dr. Abraham Wald, Dr. Edward N. Chapman, and the late Mr. W. F. C. Nelson all brought unique experience to bear upon certain aspects of the problem.

During the preparation of the book a series of conferences on economic problems was held in Colorado Springs under the auspices of the Cowles Commission. Some 200 lectures were given at these conferences and the author received many valuable suggestions both from the lectures and from informal conferences with the speakers. The effects of this unusual experience will be noted in many parts of the book.

The appraisal of the author's debt would not be complete without mention also of the help received in two other statistical laboratories, one at Indiana University and the other at Northwestern University. In the operation of these laboratories the author has been particularly indebted to Dean Fernandus Payne of Indiana University and to Professor E. J. Moulton of Northwestern University, both of whom have taken a personal interest in the work. In both laboratories many of the author's students contributed generously of their time. Colleagues in the departments of both physics and astronomy gave generously of their information at various stages of the writing of the manuscript.

Finally, but not least, the author must acknowledge his debt to the Principia Press and to its editor, Professor J. R. Kantor, who has extended in every way his cordial co-operation. The manuscript has been put into type and printed by the Dentan Printing Company of Colorado Springs, who have met all the unusual requests incidental to the production of a mathematical and statistical treatise with unfailing cheerfulness.

From these acknowledgments it will be apparent that the present work is in many respects a collaborative effort. Such virtues as the work may have are to be shared by those who have been mentioned here; unfortunately, the responsibility for the errors must be assumed only by the author himself.

H. T. DAVIS.

Northwestern University
Evanston, Illinois
November, 1941

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1949

EXTRAPOLATION, INTERPOLATION, AND SMOOTHING OF STATIONARY TIME SERIES

With Engineering Applications

by Norbert Wiener

THE M.I.T. PRESS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS

PREFACE

Largely because of the impetus gained during World War II, communication and control engineering have reached a very high level of development today. Many perhaps do not realize that the present age is ready for a significant turn in the development toward far greater heights than we have ever anticipated. The point of departure may well be the recasting and unifying of the theories of control and communication in the machine and in the animal on a statistical basis. The philosophy of this subject is contained in my book entitled *Cybernetics*.* The present monograph represents one phase of the new theory pertaining to the methods and techniques in the design of communication systems; it was first published during the war as a classified report to Section D2, National Defense Research Committee, and is now released for general use. In order to supplement the present text by less complete but simpler engineering methods two notes by Professor Norman Levinson, in which he develops some of the main ideas in a simpler mathematical form, have been added as Appendixes B and C. This material, which first appeared in the *Journal of Mathematics and Physics*, is reprinted by permission.

In the main, the mathematical developments here presented are new. However, they are along the lines suggested by A. Kolmogoroff (Interpolation und Extrapolation von stationären zufälligen Folgen, *Bulletin de l'académie des sciences de U.R.S.S.*, Ser. Math. 5, pp. 3-14, 1941; cf. also P. A. Kosulajeff, Sur les problèmes d'interpolation et d'extrapolation des suites stationnaires, *Comptes rendus de l'académie des sciences de U.R.S.S.*, Vol. 30, pp. 13-17, 1941.) An earlier note of Kolmogoroff appears in the Paris *Comptes rendus* for 1939.

To the several colleagues who have helped me by their criticism, and in particular to President Karl T. Compton, Professor H. M. James, Dr. Warren Weaver, Mr. Julian H. Bigelow, and Professor Norman Levinson, I wish to express my gratitude. Also, I wish to give credit to Mr. Gordon Raisbeck for his meticulous attention to the proofreading of this book.

Norbert Wiener

Cambridge, Massachusetts
March, 1949

*Published by The M.I.T. Press, Cambridge, Massachusetts

CONTENTS

1957

STATISTICAL ANALYSIS OF STATIONARY TIME SERIES

BY

ULF GRENANDER

DOCENT, UNIVERSITY OF STOCKHOLM

and

MURRAY ROSENBLATT

ASSOCIATE PROFESSOR OF MATHEMATICS
INDIANA UNIVERSITY

JOHN WILEY & SONS, NEW YORK
ALMQVIST & WIKSELL, STOCKHOLM

PREFACE

The purpose of this book is two-fold. It is written in the terminology of the theoretical statistician because one of our objectives is to direct his attention to an approach to time series analysis that is essentially different from most of the techniques used by time series analysts in the past. The second objective is to present a unified treatment of methods that are being used increasingly in the physical sciences and technology. We hope that the book will be of considerable interest to research workers in these fields. Keeping the first objective in mind, we have given a rigorous mathematical discussion of these new topics in time series analysis. The existing literature in time series analysis is characterized with few exceptions by a lack of precision both in conception and in the mathematical treatment of the problems dealt with. To avoid this vagueness we have devoted more space to rigorous proofs than may appear necessary to some readers, but we believe that a study of the proofs will furnish valuable clues to the practical validity of the results and be an important guide to intuition. We have tried to balance the formal proofs with intuitive remarks and comments on practical applications. While the regularity assumptions we have required in many cases may seem restrictive, appropriately interpreted they give an indication of the range in which the methods are practically valid. We have made such interpretations in the comments accompanying the formal proofs.

The reader is assumed to have a knowledge of statistics and basic probability theory equivalent to that contained in H. Cramér, *Mathematical Methods of Statistics*. The statistical techniques suggested in this monograph use concepts and relations from the theory of stochastic processes. However, we shall develop the results we need in the two first chapters. The reader who is not familiar with the mathematical techniques used in this book may find it easier to skip some of the more difficult proofs during a first reading. This is especially true with respect to Chapters 2, 4, 6 and 7. A deeper understanding of the results (and their limitations) will be obtained by returning to these proofs later.

The early attempts to formulate a theory for the statistical analysis of time series made use of a rather simple model. The observed series y_t was considered as the sum of a component m_t with no stochastic element in it,

and a disturbance x_t, where the x's are supposed to be independent and identically distributed. As an example consider the case

$$m_t = \sum_{\nu=1}^{p} A_\nu \cos (t\lambda_\nu - \varphi_\nu)$$

and let x_t be normally distributed with mean m and variance σ^2, where the constants are not specified. A typical problem would then be to estimate one or several of these parameters. A more general and flexible approach to this subject was based upon the assumption, made explicitly or not, that the underlying stochastic process was one of the so-called *finite parameter schemes*. These include the *moving average*

$$x_t = a_0 \xi_t - a_1 \xi_{t-1} + \cdots - a_p \xi_{t-p}$$

and the *autoregressive scheme*, defined as a solution of the difference equation

$$b_0 x_{t+p} - b_1 x_{t+p-1} + \cdots - b_p x_t = \xi_t.$$

Here $\{\xi_t\}$ is a sequence of identically and independently distributed stochastic variables and $\{a_\nu\}$, $\{b_\nu\}$ are constants. Modified processes with a nonrandom trigonometric or polynomial regression m_t superimposed, $y_t = x_t - m_t$, were also considered. The nonnegative integer p is called the order of the scheme.

These schemes have been important in the development of methods for the statistical analysis of time series. They have been used with a varying degree of success to describe many types of phenomena encountered in applications. From the discussion in Chapter 1 it will be apparent that by using these schemes, it is possible to approximate a large and important class of stationary processes, viz. the so-called *linear processes* (see 1.6). For this to be possible p must take large rather than small values and parameters involved in the scheme must be adjusted adequately.

During the last ten years a good deal of work has been devoted to the construction of tests, estimates and confidence intervals appropriate for these schemes. We have described a few of the more important of these results in Chapter 3. In spite of the ingenuity and great theoretical interest of some of these methods, their practical applicability seems to be limited severely by the assumption that the process is a low (usually zero, first or second) order finite parameter scheme. After surveying a good deal of the applied literature devoted to statistical analysis of time series met with in practice, we have come to the following conclusion.

Only in a few special cases (some of which will be discussed later on in this book) does it seem reasonable to assume on *a priori* grounds that the process is a low order finite parameter scheme. Referring to what has been said above, we can still approximate the process by a scheme of sufficiently high order and we can then use one of the methods developed to test the fit. This procedure is legitimate, however, *only* if we take into consideration the power of the test; usually this power will be rather small for moderate sample sizes. Hence, when we lack information concerning the structure of the process, we will have to develop methods more generally valid. If this is not possible, we should hesitate to make quantitative statistical statements, which would be based on seemingly objective methods, hiding perhaps the weak points in the argument and giving the research worker an illusory feeling of security.

At first it may seem impossible to construct methods of inference valid for the large classes of stationary processes we have in mind. Indeed, leaving the finite parameter schemes, we now deal with classes of probability distributions characterized by an infinite number of parameters. From the finite sample we obtain information concerning these parameters. This is the same problem that is encountered in the study of *nonparametric hypotheses*, although in the present context we will have to be prepared to tackle even more complex analytical difficulties.

It may be of interest to mention the two sources of ideas that we have found most useful. The first is the applied literature, especially papers dealing with statistical questions in the natural sciences and engineering. The statistician intending to do research work in this field will benefit by getting in touch with the wealth of statistical research presented in the main journals in these fields. Some of these journals are listed in our bibliography. Second, some knowledge of the modern theory of probability is indispensable, particularly the theory of stochastic processes. A complete and rigorous exposition of this subject is Doob: *Stochastic Processes*. This can be supplemented with Blanc-Lapierre and Fortet: *Théorie des Fonctions Aléatoires*, where some of the emphasis is on applications to physics.

Results have only occasionally been put in the form of theorems. This is to emphasize that they should not be considered as parts of a rigid system that can be used immediately. In the practical applications, modifications and extensions will usually be needed.

The nonparametric approach we have spoken of has been used quite recently in various fields of the physical sciences and technology although in a somewhat disguised form. On closer scrutiny, one can see that some of the basic problems dealt with in these fields are concerned with estima-

tion of the spectrum of time series, detection of signals, and other statistical problems of the type discussed in this book. The success of these methods in these concrete contexts seems to be due to the fact that in these fields people know a good deal about the structure of the random phenomena studied and so have been able to devise appropriate and relevant techniques. This can be contrasted with the rather mechanical methods of time series analysis used by theoretical statisticians in the past. The power of these new techniques is to be attributed to their nonparametric character. We have especially profited from reading the many stimulating papers in the current engineering literature. Many such papers can be found in the bibliography and we strongly advise the interested reader to examine some of these. They are especially valuable because of the problems they pose.

The basic probability model considered in this monograph is that of a stochastic process (or sequence of random variables)

$$y_t = x_t + m_t, \qquad E\, y_t = m_t, \qquad t = \cdots, -1, 0, 1, \ldots$$

with mean value $m_t = \sum_{\nu=0}^{p} c_\nu q_t^{(\nu)}$ and known regression vectors $q_t = (\ldots, q_t^{(\nu)}, \ldots)$. The residuals x_t are assumed to be a stationary stochastic process, that is, a process whose probability distribution is invariant under time shifts. This means that x_t is a stable random mechanism. In particular, it then follows that the covariance sequence

$$\operatorname{cov}(y_t, y_\tau) = E\, x_t\, x_\tau = r_{t,\tau} = r_{t-\tau}$$

depends only on the time difference $t-\tau$. Such a model fits data arising over moderate lengths of time in studies of random noise, problems in turbulence and oceanography. The model is also used in small scale investigations in meteorology. The covariances r_n are Fourier-Stieltjes coefficients

$$r_n = \int_{-\pi}^{\pi} e^{in\lambda}\, dF(\lambda)$$

of a bounded nondecreasing function $F(\lambda)$. The function $F(\lambda)$ is called the spectral distribution function of the process and knowledge of the spectrum is equivalent to knowledge of the covariance sequence. It turns out to be much more convenient statistically to deal with the spectrum rather than the covariance sequence.

The framework of the problems considered is as follows. A time series $y_1, \ldots, y_N$, a partial realization of the process $\{y_t\}$, is observed and we wish to draw inferences from the observations about the structure of the

process $\{y_t\}$. Problems of estimation and testing with respect to the regression coefficients are considered. A typical example would be that of a linear regression. Then there would be two regression vectors

$$q^{(1)} = (\ldots, 1, 1, \ldots)$$

$$q^{(2)} = (\ldots, 1, 2, \ldots, t, \ldots)$$

corresponding to the regression coefficients c_1, c_2 of the regression $m_t = c_1 + c_2 t$. Problems of estimation and testing with respect to the spectral distribution function and spectral density (derivative of the spectral distribution function) are discussed. Confidence bands for the spectral distribution function and spectral density are set up. It turns out that many of the results have an asymptotic nonparametric character, that is, many of the limit theorems (asymptotic distribution theory, etc.) obtained do not depend on the spectrum. The approach is quite different from most of the earlier work in time series analysis and is much more general in scope.

In Chapter 1 the basic probability theory required is introduced. The concepts of stationarity and spectrum are discussed and illustrated by examples drawn for the most part from physical fields.

In the second chapter the linear problems of prediction, interpolation and filtering are discussed under the assumption that the spectrum is known. Usually the spectrum is not known unless there is a good deal of prior experience in dealing with problems arising in the same experimental context. Much of the remainder of the book is concerned with the statistical estimation of the spectrum when it is not known.

In Chapter 3, the earlier work on statistical analysis of time series is surveyed. The earlier work is especially concerned with very special finite parameter models. The new techniques proposed differ in that they deal with infinite dimensional models that cover all the special models considered before and thus provide a uniform approach. The power of the new techniques lies in their great generality. The first three chapters serve as an introduction. The remaining chapters deal with the new techniques and their application.

Estimation of the spectral density is considered in Chapter 4. Two types of estimates are discussed in some detail. The first family of estimates, called spectrograph estimates in the book, are well suited for computation on a digital computer while the second class of estimates are the natural ones to build into analogue computers. The bias and asymptotic variance of these estimates are considered. It turns out that any good estimate of the spectral density is biased. The mean square error of an estimate is a

convenient measure of how good the estimate is and it is discussed in detail in the case of some special estimates.

The chapter on applications, Chapter 5, considers the model of a stationary process as it arises in several applied fields where it has been found useful. Aspects of the study of random noise, turbulence and storm-generated ocean waves are developed with this in mind.

The asymptotic distribution of a class of estimates of the spectral distribution function is developed in Chapter 6. Confidence bands are set up for the spectral distribution function and one- and two-sample tests are discussed. These results have an asymptotic nonparametric character. Remarks are made about the distribution theory of estimates of the spectral density. Examples of spectral analysis of artificially generated time series are included in this chapter.

Chapter 7 deals with regression analysis. Linear unbiased estimates of the regression coefficients are discussed. The least squares (computed under the assumption the residuals are independent) and Markov (minimum variance unbiased estimate) estimates are compared. Conditions under which the least squares estimate is as good as the Markov estimate asymptotically are given. These conditions are satisfied, for example, for polynomial or trigonometric regression. It looks as if these asymptotic results on estimation of regression coefficients are approximately valid for moderate and perhaps even small samples.

The last chapter discusses assorted problems on the maxima and zeros of time series as well as prediction when the spectrum is not known but is estimated from the time series.

The reader will notice that almost all the examples discussed in the text are chosen from the physical sciences. This is so simply for the reason that some of the most natural and successful applications of stationary stochastic processes have been in these fields.

Something should be said about the limitations of the methods presented in this monograph. As is apparent, we have studied only processes with stationary residuals. It is well known that equilibrium conditions are simpler to analyze than evolution, and the methods presented probably cannot be extended to the nonstationary case without essential changes. Futhermore, we have dealt only with discrete time, although in many of the problems we discuss, this is highly unnatural. In some cases the results can be extended to the case of a continuous time parameter (see Grenander [1] for a general outline of how this can be done) but in other cases (e.g., the problems studied in Chapter 7) unsolved problems arise, some of them of considerable analytical interest. These questions should be studied further. Very little attention is paid to vector processes although they arise in a number of important applications. Here, too, an extension seems possible and desirable (see Grenander and Rosenblatt [6] and Rosenblatt [1], [2]).

Finally, only large sample methods are considered. Because time series analysis deals with dependent observations (this reduces the amount of information obtained) and with probability distributions belonging to very wide classes, the sample size at which the asymptotic results start giving useful approximations may be fairly large. It is, of course, important to find out at what sample size such results give realistic approximations. This question deserves closer attention, perhaps via numerical methods.

If the reader is disturbed enough by these limitations to extend the methods of analysis, then this monograph will have served one of its main purposes: to stimulate research in time series analysis which will lead to practically useful and theoretically sound methods.

Each chapter of the book is divided into numbered sections. Section 6.2 refers to section 2 of Chapter 6. The numbered formulas are started *ad initium* at the beginning of each section. Formula (2) mentioned in the text of section 6.2 refers to formula (2) of that same section. Formula (6.1.2) mentioned in the text of section 6.2 refers to formula (2) of section 6.1.

Some problems have been given in the book, partly with the object of providing the reader with exercise and partly with the object of leading the reader on to derive results that supplement and extend the theory given in the text.

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1957

THE ANALYSIS OF MULTIPLE TIME-SERIES

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PREFACE

This monograph arose from research into the theory of multiple time-series. Originally, in the initial stages of research, it was planned to publish one or more papers on this subject, but it soon became clear that much more than this was required. To cover the subject in any detail would tax the space of any statistical journal over a long period and still result in a piecemeal presentation. It was consequently decided to publish the results of this research in a more readily available and complete form.

In its present form, the first of a series of short monographs published by Charles Griffin and Co., this publication is intended to provide an outline of one approach to the problems of analysing multiple time-series. It is hoped that the advanced student or research worker will find summarised here the answers to many of the problems encountered in this field, as well as a record of some of the gaps still remaining in the theory.

My thanks are due to Mr. J. Durbin and Dr. M. G. Kendall for criticism of the text and encouragement in publication, and to the Ford Foundation, whose grant jointly to the Institute of Statistics, University of North Carolina, and the Research Techniques Unit of the London School of Economics enabled this research to be carried out.

M. H. QUENOUILLE

CONTENTS

1. INTRODUCTION

1.1 Problems in the analysis of multiple time-series—The difficulties in interpreting regression or correlation analysis when applied to the investigation of time-series were first emphasized by G. U. Yule (1921, 1926, 1927). In his early fundamental papers, he points out the impossibilities of interpreting the correlations between time-series without reference to the internal structure or nature of the series.

The recognition of this fact led to a more intensive investigation of the manners in which time-series may be generated. Here again, the pioneer work of Yule in considering the stochastic development of series produced a change from the classical more determinate approach, causing statisticians (see references 1–4, 10–12, 23–25) to give much attention to the problems of recognizing, testing, and estimating the structures of time-series.

Because of the complexity of the analysis involved, the greater part of recent work has concentrated upon the analysis of individual time-series, although, in the field of economics in particular, the need for structures involving multiple time-series has become increasingly apparent. A number of discussions (see references 5, 13, 15, 22) of various aspects of the analysis of multiple time-series may be found in the literature, but there still exist numerous gaps both in the theory and between the theory and its application in practice.

It is the intention of this monograph to fill some of these gaps and to present a unified account of methods available for the analysis of multiple time-series. In doing this, the following problems are considered:

(1) In what ways may the generation of several interdependent time-series be specified and what are the correlation properties of the various specified schemes? (Chapter 2).

(2) If the correlation properties of several series are known exactly, how far is it possible to determine their method of generation? (Chapter 3).

(3) How is the answer to (2) changed if the correlations are subject to sampling variations? (Chapter 4).

(4) What are the most likely manners in which a scheme may be incorrectly specified and how will this affect the analysis? (Chapter 5).

(5) How may the parameters of any scheme be estimated (Chapter 6) and hypotheses concerning any fitted schemes tested? (Chapter 7).

(6) How well do these methods work when applied to actual practical series? (Chapter 8).

s_i	Vector giving coefficients of canonical variable for reverse scheme. Equal to $\mathbf{G}^{-1}(\mathrm{D})\mathbf{v}_i/(\mathrm{D}-\mathrm{D}_i)$; or, for Markoff schemes, row latent vector of $\mathbf{W}_1$ corresponding to D_i.
σ	Standard deviation of $\epsilon_{i,t}$.
t	Time variable indicating position of observation.
$\mathbf{t}_i$	Vector giving coefficients of canonical variable. Equal to $\mathbf{F}^{-1}(\mathrm{D})\mathbf{v}_i/(\mathrm{D}-\mathrm{D}_i)$; or, for Markoff schemes, row latent vector of $\mathbf{U}_1$ corresponding to D_i^{-1}.
u_i	Standardized coefficients, $-a_i/a_0$, of $\lvert\mathbf{F}^{-1}(\mathrm{D})\rvert = \Sigma a_i \mathrm{D}^i$.
$\mathbf{u}_i$	Column vector satisfying $\mathbf{F}^{-1}(\mathrm{D}_i)\mathbf{u}_i = 0$. For Markoff schemes, column latent vector of $\mathbf{U}_1$ corresponding to D_i^{-1}.
$\mathbf{U}_i$	Matrices of standardized coefficients, $-\mathbf{A}_0^{-1}\mathbf{A}_i$.
$\mathbf{v}_i$	Column vector satisfying $\mathbf{F}^{-1}(\mathrm{D}_i)v_i = 0$.
$\mathbf{w}_i$	Column vector satisfying $\mathbf{G}^{-1}(\mathrm{D}_i)w_i = 0$. For Markoff schemes, column latent vector of $\mathbf{W}_1$ corresponding to D_i.
$\mathbf{W}_i$	Matrices of standardized coefficients $-\mathbf{B}_0^{-1}\mathbf{B}_i$ for reverse scheme $\mathbf{G}^{-1}(\mathrm{D}) = \Sigma\, \mathbf{B}_i\mathrm{D}^{-i}$.
$x_{i,t}$	Observation in ith series at time t.
$\mathbf{x}_t$	Vector of observations, $x_{i,t}$.
$y_{i,t}$	Linear combinations of the observations taken at time t.

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1958

THE MEASUREMENT OF POWER SPECTRA

From the Point of View of Communications Engineering

By

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Preface

If this account appears to be intended principally for communications engineers it is only because an adequate understanding of how (power*) spectrum analysis works seems to demand some aspects of a communications engineering approach. (Even some of our colleagues, interested in digital computation, or in statistical techniques, rather than in communications engineering, have reluctantly come to agree with this statement.) This account is intended for all who know *what* they want to accomplish by spectral measurement and analysis (though perhaps nothing of *how* to accomplish it), and are concerned with how to do it, or with how to think about doing it, or with why it should be done in one way rather than another. Considerable mathematical detail is given, but as a guide and background to practice, rather than either for its own sake or for the sake of rigor.

This account was written as an extended journal article, and not as an introduction to the beauties of spectral analysis. Thus, there is no discussion of why one might want to estimate (power) spectra, and no catalog of the wondrous results thus obtained. We cannot refrain, however, from quoting one wondrous result (from a letter from Walter E. Munk to one of the authors): ". . . we were able to discover in the general wave record a very weak low-frequency peak which would surely have escaped our attention without spectral analysis. This peak, it

* The use of the adjective "power" comes from a habit of language of communications engineers which we need not discuss here.

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turns out, is almost certainly due to a swell from the Indian Ocean, 10,000 miles distant. Physical dimensions are: 1 mm high, a kilometer long."

Except rather summarily, later in this preface, we do not treat and distinguish the many sorts of Fourier methods which are available. All in all, our account has quite limited aims.

Groups of Readers

We can, we feel, recognize four large groups of readers, as far as attitudes and backgrounds are concerned. There will be a group of communications engineers, to whom the sort of Fourier analysis we use will be relatively familiar, but whose statistical background may be quite limited. There will be a group of digital computermen whose clients have, or threaten to have, data on which spectral calculations will be required. Their background in the communications engineering approach is quite likely to be limited, as will be, perhaps, their background in statistics. There will be a group of statisticians without special background in this sort of time series analysis, to whom the communications engineering approach is very strange, and who wonder why "cosines have anything to do with things that are not periodic." There will, we hope, be a fourth group of readers who have, or know how to get, data they wish to analyze, and who may well lack background in all three fields: communications engineering, computer techniques, and statistics.

We say to the members of each group: "Though it may seem to you that it is not so, we have tried to consider you in writing this account. Matters are not simple, but we could have easily made them more complex." If we had been willing to try to write this account for some one group, or, more particularly, for some one part of one group, we should perhaps have been able to simplify its organization. It will be important for each reader to understand the structure actually used in this account, and to recognize that material complementary to that in Section x is to be sought in Section B.x. That when Section y expresses mainly statistical aspects, the communications engineering aspects are likely to be found in Section B.y. And *vice versa*. That, if Section z expresses the more formal aspects of a topic, the more verbal aspects are likely to be found in Section B.z. And *vice versa*.

Questions of Orientation

There are three sorts of questions which are likely to bother many readers at the beginning — questions to which answers are needed to set these readers on the right road.

(1) What are the different kinds of Fourier analysis? How are they to be distinguished? (What kinds of data are appropriately treated by these various methods? How can I tell which to use?)
(2) Why do cosines appear in the analysis of aperiodic phenomena? Isn't this unnatural?
(3) What other sorts of spectral analysis are there? How are they related to those discussed here?

We shall try to answer these questions briefly in the remainder of this preface.

The Different Kinds of Fourier Analysis

There are four ways of distinguishing among Fourier techniques, corresponding to questions which can be answered in a "yes or no" fashion. The first two are:

(1) Is the phenomenon treated periodic? (Or, equivalently, are all of the frequencies involved in the phenomenon integral multiples of a single frequency?)
(2) If the phenomenon is aperiodic, are the frequencies involved discrete (distinct) or continuous?

When we recognize that measured values will reflect not only the phenomenon studied but also measurement errors, and, usually, other sources of fluctuation, we see that the data will always be aperiodic. Thus, to the combination of these two questions there are but three essentially distinct answers, which may be phrased as follows:

(a) Phenomenon aperiodic, frequencies continuous, data aperiodic,
(b) Phenomenon aperiodic (but "almost periodic" in the mathematical sense), frequencies discrete but not harmonically related, data aperiodic.
(c) Phenomenon periodic, frequencies discrete and harmonically related, data aperiodic.

A third way of distinguishing among Fourier techniques corresponds to the question

(3) Are the times involved discrete (and equi-spaced) or continuous? As explained in Sections 13 and 14 for one case, the practical distinction between discrete and continuous time is negligible (although there may be differences of mathematical theory). The answer to this question is usually not important.

A fourth way of distinguishing among Fourier techniques corresponds to the question

(4) Are the data analyzed thought of as unique, or as a statistical sample?

The answer to this question is exceedingly important, even though it might not seem that this should be the case.

Either of the answers to (4) can be combined with answer (a), or (b), or (c) to (1) and (2), making six possible such combinations. We have thus six pairs of kinds of Fourier analysis, the two kinds in a pair differing only in whether time is continuous or discrete.

In three of these pairs we wish to treat the data as a statistical sample. The text discusses in detail the pair corresponding to answer (a) to questions (1) and (2).

No especial development of theory or practice seems to have yet occurred for the other two statistical-sample pairs. Data which one might possibly suppose to fall under one of these pairs are usually analyzed by the methods described in the text, just as if they fell under the first pair. This practice makes more sense when we realize that no finite amount of data can determine whether a function is aperiodic, almost periodic, or periodic. A seemingly aperiodic function *might* start to repeat exactly with a period of, say, three times the length of the available record. A seemingly periodic function may suddenly cease repeating at a point just outside the available record. And the inevitable concealment of fine detail by measurement noise (by fluctuations and errors of measurement) blurs such distinctions even more completely.

There remain three pairs of kinds of Fourier analysis, in all of which we wish to treat the given data as coming from a unique function. Probably the simplest pair involves periodic phenomena, arising, for example, in connection with seasonal (or daily) fluctuations in climate, weather, or economic series, or in connection with physical phenomena related to an angular orientation (in a plane). In practice a very important distinction must be made between cases in which

(i) the periodic phenomenon dominates the data, irregularities (fluctuations or errors) of measurement or recording being negligible,

(ii) the periodic phenomenon must be sought among substantial irregularities, and

(iii) the periodic phenomenon is to be separated from both irregularities and slow trends.

In the uncomplicated case (i), almost any method of analysis will be satisfactory if its arithmetic is kept simple. Such methods are discussed, often under the heading of "empirical harmonic analysis", in books on numerical computation (especially in Theodore R. Running's *Empirical Formulas*).

Methods of analysis for case (ii) may have to be more subtle. The outstanding reference is Chapter XVI, "Periodicities and Harmonic Analysis in Geophysics" by Julius Bartels on pp. 545–605 (Volume 2) of *Geomagnetism* by Sidney Chapman and Julius Bartels, Oxford University Press (1940, second edition 1951). (Consideration should also be given to the techniques developed by the Labroustes; see references 41, 42, 43 in the text.)

In case (ii) situations, the use of methods appropriate to case (i) situations may be very dangerous.

For case (iii) we have a choice. Most case (ii) methods are applicable with but little change and provide relatively reliable answers. Rather specific methods are widely used in economics, where usually, and probably wisely, the periodic parts are not expressed in Fourier form. Such methods are discussed in many texts in economic statistics.

(The most complete summary of the classical Fourier series and periodogram approaches seems to be that of Karl Stumpf, *Grundlagen und Methoden der Periodenforschung*, Springer, Berlin, 1937; Edwards, Ann Arbor, 1945, who also provided convenient tables, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung*, Springer, Berlin, 1939. Note that Stumpf's first graph, on page 45 of *Grundlagen und Methoden der Periodenforschung*, illustrates aliasing. A number of related topics are treated in Cornelius Lanczos, *Applied Analysis*, Prentice-Hall, Englewood Cliffs, 1956).

The next pair involves almost periodic phenomena, which are of frequent occurrence in geophysics (lunar and solar tides, etc.) and occurs, as an approximation, in such astronomical problems as variable-star light curves. For the *rare* instances in which the periodic terms are well separated in intensity *and* the irregular background is small, excellent use can be made of the treatment in Whittaker and Robinson's *Calculus of Observations*. The best reference, however, is the Bartels chapter already mentioned.

There remains one pair of kinds of Fourier analysis, in which the data is thought of as unique and aperiodic. The only natural excuse for bringing cosines into such situations is a desire to study the effect of one or more linear time-invariant transformations. For a theoretical problem, the theory and practice of Fourier integrals stand ready at hand. For a data analysis problem, very little help is available, since the Fourier integral transform always involves more parameters than we can hope to estimate from any finite amount of data. In practice, then, we must approximate this pair of kinds of analysis by some other pair. If we treat the data as a sample rather than unique, we come to the pair treated in the text. If we treat the phenomenon as periodic, or almost periodic, rather than aperiodic, we come to one of the pairs just discussed above. Some such shift is essential.

In practical data analysis, of course, we must always be prepared to approximate wisely. A pressure wave representing human speech, for example, is not precisely periodic, but many short stretches of it are so nearly periodic that a periodic analysis is often very fruitful.

Equally important with a willingness to approximate (far enough, but not too far) is a willingness to take a statistical view whenever the data represents at most "something like" the situation of eventual concern. (It is indeed very rare in practice that the data under analysis represent the identical situation about which conclusions are to be drawn.)

Why Cosines for Aperiodic Phenomena?

It is surely natural, once a statistical view is taken, to study averages, first of linear expressions and then of quadratic expressions. It is also natural to begin by studying situations which are independent of clock-setting in the sense that whatever started to happen at one (clock) time could equally likely have started at any other (clock) time. (Such situations are termed stationary, in technical language.) The average values of all linear expressions can then be described by one number (because the average values of all individual values are all the same).

Average values of quadratic expressions are more complicated. To describe them we have to specify, not just a single number, but a single function. (If time is discrete, this function can often be specified by an infinite sequence of numbers.) There are many choices of such *bases*, each of which will serve to describe the average values of all quadratic expressions. All bases contain the same information, but some code it

in more secret (hard to read) ways than others. The power spectrum itself, the autocorrelation function (or, better, the autocovariance function), and the average values of $[X(t + \tau) - X(t)]^2$ (as a function of τ), are only three among many choices. *Exact* and *complete* knowledge of the values of any choice, of course, determines the values of each of the others. However, approximate knowledge of the power spectrum does not determine the others very well, nor does approximate knowledge of either function of lag determine the power spectrum very well. Frequency and time correspond to very different forms of coding — small, obvious errors in terms of either one may correspond to a nearly nonsensical message in terms of the other.

It is natural for the statistician to say "So what, I wanted to use the autocovariance function anyway. All that you have told me merely encourages me to forget cosines and power spectra." To this view, there are two counter-arguments of importance, one general, the other statistical.

Whether it be called a stochastic process, a noise, or a signal, we are very likely to want to know what will happen when it is subjected to a time-invariant linear transformation; when, otherwise expressed, it is passed through a linear, time-invariant filter. If we are thinking of it statistically, we are almost certainly concerned with averages of simple expressions for the output. (The natural basis for the averages of linear expressions, the average value at any time, will, of course, be multiplied by the voltage transfer function of the filter, evaluated at zero frequency.) The output values, expressed in terms of any basis for the averages of quadratic expressions, will be linearly expressible in terms of the corresponding input values, and may be regarded as the result of a linear transformation on the values describing the input. However, some linear transformations are simpler than others. Surely the diagonal transformations are the simplest. The transformation expressing the effect of any filter at all, as seen in terms of the values appearing in the *power spectrum* description, is diagonal. The transformations corresponding to the various lag functions are almost never diagonal, no matter how simple the filter.

Thus, if we desire convenience in handling average values of quadratic (or general second-degree) expressions, (i) we will use some basis purely for compactness, (ii) among all bases we will select the power spectrum basis because of its much simpler transformation properties. Simplicity alone forces us to consider cosines in connection with aperiodic stochastic processes. This is the general reason.

The statistical reason lies in the joint behavior of sampling fluctuations. Suitably spaced estimates of (smoothed) power spectral density fluctuate in a rather thoroughly independent manner. On the other hand, estimates of functions of lag (be they autocorrelations, autocovariances, or average values of $[X(t+\tau) - X(t)]^2$) have fluctuations which are so far from independence as to frequently fool almost anyone who examines tables or graphs of their values. From this point of view, also, the power spectrum has important advantages. All in all, there is little hope of escaping cosines.

Related Types of Spectral Analysis

Estimation of spectra of single functions or of single series (which is the only subject discussed in the text) is not the only form of spectral analysis. When two or more concurrent series are available (originating from related phenomena), cross-spectra can be calculated, two for each pair of series. (The theory has been discussed by Goodman, Ref. 1 in the text.) Such analyses provide opportunities for studying such problems as: the behavior of certain linear transformations (by studying cross-spectra between input and output); the two-dimensional structure of atmospheric turbulence (by studying cross-spectra between vertical and horizontal components of wind velocity); the direction of arrival of ocean waves (by studying cross-spectra of records at different locations); and the gust structure of the atmosphere well away from the ground (by using cross-spectra between airplane accelerations and control positions to determine what these accelerations would have been with the controls locked). Besides the rather technical account by Goodman, brief introductions are given by Press and Tukey (cited in the bibliography) and in a paper by Tukey "An Introduction to the Measurement of Noise Spectra" which appears in *Surveys in Probability and Statistics*, Almquist and Wiksell, Stockholm, 1958.

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Time Series Analysis

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Author's Preface

The analysis of time series is usually studied after a course in the classical part of statistical theory has been completed. For that reason this book assumes a knowledge of mathematics and statistics approximately up to the level of Cramer's text *Mathematical Methods of Statistics*. In the author's experience, even then, the student has difficulty, especially with the spectral theory. This is due to the unavoidable introduction of infinite dimensional vector spaces. A heuristic introduction to the spectral theory has therefore been given early in the first chapter, which reduces the discussion to one concerning finite dimensional vector spaces. Proper proofs of the main theorems of the spectral theory are given later in the chapter and in the appendix, and throughout the book this is done for most of the results which are cited.

The remaining chapters deal with statistical inference. The most general model considered is a univariate time series consisting of a time dependent mean plus a stationary component. A discussion of multivariate series was not possible because of the limited size of the book. The restriction to stationarity is necessary since a theory for evolutive series hardly exists, and almost all methods in use at present depend upon the reduction of the series to something approaching stationary form by elementary statistical devices.

This book was written while the author was a fellow at the Australian National University and arose out of a series of lectures delivered at the University of Western Australia. The author has learned much about this subject from discussion with Dr G. S. Watson and Dr P. Whittle, and to them, and most especially to Professor P. A. P. Moran, who introduced him to the subject, he expresses his gratitude.

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the function $g_\theta(\lambda)$, which is $g(\lambda)$ in the range $[-\pi, \theta]$ and zero outside of it (which evidently lies in $\mathscr{M}$), may be approximated uniformly by a function of the type $\sum_k g_\theta(\lambda_k)\{e_{\lambda_k}(\lambda) - e_{\lambda_{k-1}}(\lambda)\}$ it is evident that the proof of the spectral representation formula for x_t carries over directly to give (4) its precise meaning.

Finally since $g(\lambda_k)\{e_{\lambda_k}(\lambda) - e_{\lambda_{k-1}}(\lambda)\}$ may be made uniformly close to $g_{\lambda_k}(\lambda) - g_{\lambda_{k-1}}(\lambda)$ by choosing a fine enough decomposition of $[-\pi, \pi]$, it follows that

$$\sum_k e^{it\lambda_k}\{e_{\lambda_k}(\lambda) - e_{\lambda_{k-1}}(\lambda)\} - \sum_k e^{it\lambda_k} g(\lambda_k)^{-1}\{g_{\lambda_k}(\lambda) - g_{\lambda_{k-1}}(\lambda)\}$$

converges, with n, to zero, in the mean with weighting $dF(\lambda)$. This implies in turn that

$$x_t = \int_{-\pi}^{\pi} e^{it\lambda} g(\lambda)^{-1}\, dy(\lambda)$$

which justifies the substitutions made in section (I.3).

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1962

AN INTRODUCTION TO THE THEORY OF STATIONARY RANDOM FUNCTIONS

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Revised English Edition
Translated and Edited by
Richard A. Silverman

1962
PRENTICE-HALL, INC.
Englewood Cliffs, New Jersey

AUTHOR'S PREFACE TO THE RUSSIAN EDITION

The basic aim of this monograph is to give a simple and at the same time mathematically rigorous treatment of the problem of extrapolating and filtering stationary random functions (both sequences and processes). The material stems from two sets of lectures given at Moscow State University, one for a group of graduate students in the Mechanics and Mathematics Department, and another for a seminar under the direction of E. B. Dynkin. In order to keep the presentation as elementary as possible, emphasis has been put on the simplest special case, where the spectral density is a rational function (in $e^{i\lambda}$ for sequences, and in λ for processes). However, Chapter 8 contains a brief survey of the results obtained by A. N. Kolmogorov in his deep study of the general case.

Part 1 of the book is devoted to a rather complete presentation of the spectral theory of stationary random functions. This sophisticated theory, originating in A. Y. Khinchin's paper in the *Mathematische Annalen* (vol. 109, p. 604, 1934), is now the basis for almost all research on the subject. Part 1 should be of independent interest, aside from its connection with the theory of extrapolation and filtering, presented in Part 2.

The reader is assumed to have at his command little more than the rudiments of probability theory and complex variable theory. However, to understand Part 2, he will need to know the most elementary facts about the geometry of Hilbert space. The reader who is not familiar with this material may have to take certain statements on faith.

Frequent discussions with A. N. Kolmogorov have had a considerable influence on all my work in the field of

random functions, and I have also profited greatly from conversations with A. M. Obukhov. While writing this monograph, I have received substantial help from A. S. Monin, which has enabled me to finish the work much sooner than would otherwise have been possible. A. N. Kolmogorov and A. M. Obukhov have both read the manuscript, making many valuable suggestions. I am delighted to take this occasion to express my sincere gratitude to these three colleagues.

1952 A. M. Y

AUTHOR'S PREFACE TO THE REVISED ENGLISH EDITION

The original version of this monograph was published a decade ago as a long review article in the Russian journal *Uspekhi Matematicheskikh Nauk* (vol. 7, no. 5, 1952). At that time, no book specifically devoted to the mathematical theory of stationary random functions was available, either in Russian or in any other language, and it was my intention to remedy, at least partially, this gap in the literature. I also wanted to popularize, for as large an audience as possible, the theory of linear extrapolation and filtering of stationary sequences and processes, due to A. N. Kolmogorov and N. Wiener, a theory which is both of intrinsic mathematical interest and of great practical importance. This task seemed to me even more worthwhile for another reason: On the one hand, Kolmogorov's basic paper, containing complete proofs of his profound results in this field, had been published in a journal with a rather limited circulation (*Bulletin Moscow State University*, vol. 2, no. 6, 1941), while the

methods used in the paper are complicated and accessible only to students with an extensive mathematical background. On the other hand, Wiener's celebrated report, written during the war, had just appeared in book form at about this time, and it immediately acquired a reputation among engineers of being extraordinarily abstruse, whereas most mathematicians, unaccustomed to its heuristic level of rigor and engineering terminology, had great difficulty in understanding it. However, I had found that it was quite feasible to give a simple and entirely rigorous treatment of the problem of extrapolating and filtering stationary random functions, for the case of rational spectral densities. In fact, a recent article by S. Darlington (*Bell System Technical Journal*, vol. 37, p. 1221, 1958) advocates the same method as the most suitable approach for engineers encountering the subject for the first time.

During the last decade, a number of interesting books on the theory of random functions have been published, many of which are cited in the bibliographies at the end of this volume. Nevertheless, it seems to me that my book, which, without sacrificing mathematical rigor, stresses the physical meaning of results rather than delving into logical subtleties, will be of interest to beginning mathematicians, as well as to physicists and engineers who actually deal with random functions in practice. However, despite the fact that the book treats only elementary aspects of the theory, the presentation of some topics in the Russian original was in danger of becoming out of date, because of subsequent advances in the field, on all levels. Therefore, the translator, Dr. R. A. Silverman, decided to "rejuvenate" the book, by adding new material in keeping with the present development of the subject (especially in Chapters 3 and 8, which have a survey character), by enlarging and modernizing the Bibliography, by supplying numerous footnotes containing explanatory comments and references to recent work, etc. I myself have also made numerous improvements and additions, working from a copy of the manuscript sent to me by the translator in ample time for revision. Moreover, I have read through all the galley proofs, making sure that I approved of the final version. Thus,

in my opinion, the English edition of the book reflects the contemporary state of research in the theory of stationary random functions.

Of course, in a book of this size, it has been impossible to discuss all aspects of a theory which has progressed so rapidly, and it is inevitable that some important results are mentioned only in passing, or are not touched upon at all. However, it seemed to me that failure to include a discussion of the recent theory of generalized random processes would have been regrettable, since this theory greatly simplifies the treatment of some important topics, e.g., random processes with stationary increments. Moreover, the concept of a generalized random process is very "physical," and hence quite accessible to those readers for whom the book is primarily intended. Thus, I thought it advisable to write an elementary exposition of the theory of generalized random processes, expressly for the English edition, and I accordingly sent the translator an outline which he expanded into the first appendix. In addition, Dr. D. B. Lowdenslager contributed a second appendix, containing a brief survey of some recent developments (mostly pertaining to extrapolation and filtering of multidimensional stationary random functions), with appropriate supplementary references. This appendix should also enhance the value of the book.

I am pleased that my book is appearing in a special series of Russian translations, since this will serve to bring it to the attention of many new readers. I would like to express my appreciation to the Prentice-Hall Publishing Company, to Dr. D. B. Lowdenslager, and especially to Dr. R. A. Silverman for his careful translation and painstaking effort to eliminate typographical errors in the Russian original, for helping me write Appendix I, and in general, for doing everything in his power to improve the English edition of the book.

1962 A. M. Y.

TRANSLATOR'S PREFACE

The present volume, the fourth in a new series of Russian translations under my editorship, has long been regarded as a classic presentation of a subject of great theoretical and practical interest. It has been a great pleasure to work with Professor Yaglom in preparing a revised English edition of his book. I would like to thank him for his kind words above, and for his indefatigable cooperation during a four-month period of heavy correspondence. Professor Yaglom has already described both his aims in writing the monograph, and the ways in which the English edition differs from the Russian original. Thus, at this point, I would only like to call attention to a few stylistic matters:

1. The system of references in "letter-number form" used in the main Bibliography is almost self-explanatory. For example, K9 refers to the ninth paper (or book) whose (first) author's surname begins with the letter K, where the entire Bibliography is arranged in alphabetical order, and in chronological order as well, whenever there are several papers by the same author.

2. The main Bibliography contains only items cited in the text, and the Supplementary Bibliography contains only items cited in the two appendices. It is not claimed that the bibliographies are exhaustive, especially as regards papers on applied topics.

3. Sections marked by asterisks, and also individual passages lying between asterisks, contain more advanced or detailed material, which can be omitted without loss of continuity. However, no attempt has been made to indicate such passages in Chapter 8, which is essentially a survey chapter.

1962 R. A. S.

CONTENTS

1963

Robert Goodell Brown

Arthur D. Little, Inc.

Smoothing, Forecasting and Prediction of Discrete Time Series

Prentice-Hall, Inc. Englewood Cliffs, N.J.

Preface

"These hieroglyphics have evidently a meaning. If it is a purely arbitrary one, it may be impossible for us to solve it. If, on the other hand, it is systematic, I have no doubt that we shall get to the bottom of it."

Adventure of the Dancing Men

In recent years, the rapid rise of technology in industry and in government has created both the need and the means for effective use of high-speed, internally programed, digital computers. As computers are employed more and more to carry out the routine data-processing functions for a business, or for the Defense Department, there is a stronger and stronger pressure to develop means for handling all the steps in the problem routinely. No intermediate print-outs, that delay the processing, should be required. The information should be "untouched by human hands," from the original input to final output. As Sir Walter Scott wrote* in 1830, "The times have changed in nothing more than in the rapid conveyance of intelligence and communication."

* *The Heart of Midlothian.*

In many applications, computers were first employed solely for routine data processing: the preparation of payrolls, maintenance of inventory lists, posting production progress, and recording of stock status. These applications require that information be printed out, either in summary or in detail, for someone to look at and make a decision for some sort of action. Operations research studies have made it possible to pass from the use of computers for bookkeeping to their use for control: production scheduling, stock replenishment, capital budgeting, air defense, and fire control systems. Each of these applications requires an estimate of what will happen in the future. Men with sufficient skill, judgment, and experience can do a reasonable job of predicting the future, given enough time and information. These men are frequently reluctant to admit that a computer can be taught to forecast well. A reluctance to exploit the speed, capacity, and flexibility of a computer has been noticed among pilots who were skeptical of bombsights and fire-control systems, and among stock analysts when first faced with a modern integrated data-processing system.

A computed forecast may not always be more accurate than a human prediction. It can be obtained so much faster and so much more cheaply, however, that it may be advantageous to sacrifice some accuracy if necessary. More frequently, the machine's forecasts, on the average, are more accurate than the conventional human predictions, by a measurable amount.

The scope of this book is limited to the current state of the art of programming digital computers to compute forecasts of discrete time series. The analysis of the entire control system that makes use of the forecast is beyond this book. We are concerned primarily with what the control engineers call the *open-loop* characteristics of one box that accepts current observations of the time series and delivers a forecast of the probability distribution from which future observations will be drawn. The analysis and development of these techniques has been kept sufficiently general that they can be applied to a very wide variety of integrated control systems.

Objective forecasts, of the type that can be programed for a computer, are dependable and unemotional: their response to changes in the external environment can be studied in advance and systems can be designed from these studies; the computations are consistent and therefore controllable across a wide variety of problems. Since perhaps 80 to 95 per cent of the problems encountered are quite routine, they can be handled by the computer. Thus, the analyst has from five to twenty times as much effort available to spend on the exceptions that really do require his skill, judgment, and experience.

In 1959, the McGraw-Hill Book Company published my *Statistical Forecasting for Inventory Control* which reported the state of the art at that time, with special reference to inventory control applications. Research in the problems of statistical forecasting has proceeded steadily since then, and much more powerful methods have been developed, particularly in the description of a time series by much more general classes of functions. Of especial interest to businesses with seasonal demand should be the class of trigonometric functions that make it possible to describe any cyclical process accurately and easily.

The organization of this book has been something of a problem to me. The objective of our research has been the development of practical methods that can be applied to real problems in the government and in business. Therefore, the primary results should be presented in a "How to" fashion that the men who are directly concerned with the problems can understand. On the other hand, these results stem from some intricate reasoning that can be carried out accurately only in the language of mathematics, and not everyone can speak that language fluently. The better the reader understands where the results came from, the surer he is of applying them correctly.

As each new topic is introduced, there is a non-technical summary of the major results to be obtained. Where possible, I have given a plausible argument that is intended to make the results seem reasonable. Numerical examples are also used to make a point clear. There are work sheets so that you can work out additional examples by hand. These work sheets can easily be converted to computer programs. With diligence and patience, almost anyone should be able to get a sufficient understanding of the procedures to apply them and to get the correct answer.

Complete mathematical derivations are also given, with the formulas and tables necessary to extend the range of coverage beyond the problems that can be illustrated by examples. A standard college background in mathematics should be sufficient for an understanding of the principal results in the more technical sections. Only a professional mathematician, however, will fully appreciate the results and be able to extend them to new areas. The topics of interest to the mathematician primarily are written in more technical language than that used in discussing the problems and procedures for dealing with them. Such sections also assume that the reader has a more advanced background. The text is larded with exercises to help stimulate thought about the problems.

Whenever you feel that you are getting in over your head, skim the material for a while, until the discussion becomes less theoretical. Occasionally, you will find that you must go back and work on an earlier section to get the material needed as a foundation for something later. The classroom teacher will find it advisable to skip some material and to cover the remainder in a sequence that suits the needs of the class.

A teaching sequence that has been successful is first to cover one complete, although elementary system based on Chapters 1, 4, 8, 22, 23, and 25,

followed by units on data (Chapters 2 and 3, Appendix A); time series models and the characteristics of smoothing systems (Chapters 4, 9, 10, 11 and Appendix B); general exponential smoothing and forecasting (Chapters 12, 15, and 16), Error Measurement (Chapters 19 and 20). At the end of the course, the class can take up special topics, such as probability models (Chapters 5, 13, and 17), the direct forecasts of Chapter 18, and optimum linear filters (Chapter 21).

A number of concepts are used in this book. The reader who is familiar with these concepts in other areas should have no difficulty. The reader who is aware that he is going to encounter new concepts should have no difficulty in getting a sufficient understanding of them from the context. Some of the following concepts are trivial, some are quite deep. I have found that they have created problems for people who are not aware that they *are* new to them: probability distribution, functions, least squares, simulations, systems analysis and design, transforms, and so on.

A number of formal manipulative techniques are also used freely in this book. The reader who wants to go deeply into the developments discussed here should have a good working knowledge of algebra, the calculus, mathematical statistics, and matrix algebra. If the reader knows something about computer programming, he will more fully appreciate the reasons for some of the development. It is not necessary, however, that he be a professional programer. Two techniques are used in some depth: *z*-transforms and the regression analysis. Since these are not generally understood clearly, a brief review of the essential developments is included in the appendices.

Each of the six major parts of this book thoroughly discusses the alternative choices facing the systems designer in (1) deciding on the sources of the data to be used; (2) the model to represent the data; (3) the smoothing technique to estimate values for the model from the current data; (4) the forecast obtained from the model; (5) the measurement of error in the forecasts; (6) the applications of the forecast and error measurement to a particular decision problem. For a particular application, many of these alternatives may not be relevant. The teacher may therefore organize the material around a single thread that selects one alternative in each section, leading to a particular system for forecasting. When that system is well understood, he may then go back over some of the alternatives that might have been considered at each stage of the design.

Much of the work leading to this book has been carried out in the course of industrial assignments by the Operations Research Section at Arthur D. Little, Inc. During 1961 and 1962, the Bureau of Supplies and Accounts has supported basic mathematical research in the techniques of forecasting under Contract Nonr-3406(00). I should like to express my appreciation for the support and encouragement offered by Captain Ed Scofield, SC, USN, Commander Herb Mills, SC, USN, and Messrs. Randy Simpson and Jim Prichard.

I gratefully acknowledge the help of many colleagues and acquaintances who have made numerous comments and suggestions. Especial mention must be made of Professor G. E. P. Box of the University of Wisconsin, Professor Ronald A. Howard of MIT, Professor Sebastian Littauer of Columbia University, Mr. Warren Briggs of The RAND Corporation, and Messrs. Jim Loughney and Peter Woitach of IBM's Systems Research Institute. Dr. Robert Barringer, Michel Carré, Gordon Crook, Dr. James Dobbie, Dr. Ernest Foernzler, Frank Hulswit, Mrs. Elizabeth Hutton, Dr. Richard Meyer, Lawrence Parker, Dr. Stefan Peters, Peter Strong, and Miss Joan Sullivan at Arthur D. Little, Inc., have contributed greatly in suggesting novel ideas, developing proofs, and carrying out detailed examples. Miss Norma Moulton has coped admirably with the unenviable task of typing and retyping the manuscript.

Chris Kentera, Robert Carola and Norm Stanton of Prentice-Hall have made the work of publishing this book seem, to the author at least, a simple task and a great joy.

The quotations at the beginning of the various sections are taken from the Sherlock Holmes stories by Sir Arthur Conan Doyle, and are used by permission of Sir Arthur Conan Doyle's estate.

ROBERT G. BROWN

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Appendix

Work Sheets

1 Introduction

"The emotional qualities are antagonistic to clear reasoning."

The Sign of the Four

You're driving an automobile. You glance at the gasoline gauge and decide to drive on past a service station. In that decision you have weighed your estimate of when you'll next pass another station and your estimate of the rate of consumption of gasoline, and decided that your present supply is greater than the consumption until the next replenishment opportunity. You can do these computations quite subjectively and can afford to carry a quarter tank of gasoline that is never used, just to be quite sure that even if you make a reasonable error you won't get into trouble.

A businessman may have to make similar decisions for each of thousands of items he keeps in stock. It is vitally important for him to be able to reduce his investment in unused stocks. Objective computations of the probabilities of consuming stock at various rates and objective computa-

The development of any new idea depends heavily on the ideas that preceded it. Wherever a remark in this book stemmed from a specific reference, a footnote gives the necessary details. This bibliography is intended as a general reading list for the student who wants to recapitulate for himself the ontogeny of current techniques of statistical forecasting.

Books marked with an asterisk (*) have particularly good bibliographies of journal articles.

I am deeply indebted to Mrs. Elizabeth J. Hutton for her patience, industry, and ingenuity in assembling these references.

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Glossary of Symbols

$\mathbf{a}$	vector of coefficients, 163
$\hat{a}_T$	estimated coefficient at time T, 61
$\hat{b}_T$	estimated trend at time T, 61
e	exponential base (2.71828 . . .), 120, 437
$e_\tau(t)$	forecast error for a lead time τ, 291
$E(t > k)$	partial expectation, 371
$\mathcal{E}[x]$	expected value of x, 101
$f(k)$	partial expectation, 373
$\mathbf{f}(t)$	vector of fitting functions, 162
$f_i(t)$	the ith fitting function at time t, 76
F	matrix of weighted fitting functions, 163
$F(k)$	probability, 377
$F_x(j\omega)$	Fourier transform, 276
$\mathfrak{F}$	matrix of fitting functions, 162
$\mathbf{g}(t)$	data vector, 164
$G(\omega)$	gain, 397
h_n	impulse response, 407
$\mathbf{h}$	smoothing vector, 175
$H(z)$	transfer function, 146
L	transition matrix, 165
M_t	moving average, 98
$M_t^{[2]}$	double moving average, 127
MSE	mean square error, 119
$\mathbf{p}(t)$	probability vector, 201
$\mathbf{P}$	probability vector, 83
$P(t)$	probability of arriving home before t, 55
$P_x(\omega)$	power spectrum, 398
$R_{xx}(\tau)$	autocovariance, 47
$S_t(x)$	exponential smoothing operator, 101
$S_t^{[p]}(x)$	pth order exponential smoothing, 128
t	time index in general
T	time of the most recent observation
$\binom{t}{k} = \frac{t!}{(t-k)!k!}$	binomial coefficient, 66
$\mathbf{u}$	unit vector, 200
V	variance-covariance matrix, 229
W	weighting matrix, 391
W_T	weighted moving average, 125
x_t or $x(t)$	observed data, 25
$\hat{x}_\tau(T)$ or $\hat{x}(T+\tau)$	forecast made at time T for a lead time τ, 272
$\hat{x}_t^{(n)}$	estimate of the nth derivative, 132
$\overline{x_t x_{t-k}}$	average lagged product, 119
$Y(T)$	sum of the forecast errors, 290
α	smoothing constant, 101
$\beta = 1 - \alpha$	discount factor, 92
$\delta(t)$	Dirac delta function, impulse, 112
$\Delta(t)$	mean absolute deviation, 283
Δ^n	nth difference, 63
ϵ_t	noise sample, 60
θ	phase angle, 116
$\theta(x)$	probability distribution, 253
$\xi(t)$	process generating observed data, 60
$\rho(k)$	autocorrelation coefficient, 394
σ^2	variance, 47
σ_τ	standard deviation of forecast errors over the lead time, 365
$\Psi(\omega)$	frequency response function, 397

1963

PREDICTION AND REGULATION

by Linear Least-Square Methods

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PREFACE

The prediction of a random process in time has been studied as a topic in probability, and as a technique in the fields of communication and control: this book is an attempt at something intermediate in character.

The classical probabilistic treatment is to consider the pure prediction problem (i.e. prediction of the process which is actually observed) for a general stationary process; it culminates in a derivation of the conditions (formulae (2.7.9), (2.7.10)) for linear determinism. Several good treatments of this type already exist (Doob, 1953; Grenander and Rosenblatt, 1957); to write another would be pointless. Furthermore, from the applied point of view this approach is at once too general and too special: one will not be interested in the general stationary process, but one may well have to deal with processes which are not stationary at all, and the concept of pure prediction very soon proves over-narrow.

There is a certain amount of what I believe to be new material in the book, although in rather dispersed form. The sections on accumulated processes and multivariate processes (Chapters 8 and 9, cf Yaglom (1955, 1960)), prediction from finite samples (Chapter 7) and regulation (Chapter 10) contain some such results, which, if not new, are at least novel. The important concepts and techniques of linear least-square regulation theory are, of course, due to G. C. Newton (1952, 1957), but trying to integrate these into the book, I found that quite a number of interesting new points arose.

It was intriguing to find in the course of writing that there were two themes that recurred quite spontaneously. One of these was the characterisation of a predictor (or regulator) by the processes for which it gave an exact result. The other was the continual dualism between the time-domain and frequency-domain approaches, typified by the methods of Kolmogorov and of Wiener respectively. This dualism persisted right through to the section on regulation, where the obvious Wiener-Hopf methods had as analogue in the time-domain the fascinating idea of certainty equivalence.

Regrets which I have are that so little attention is paid to processes with irrational spectral density function and to processes in several dimensions; also that, outside the chapter on regulation, there are so few detailed discussions of special cases of the degree of complexity that would be met in practice. However, if one wishes to retain the advantages of brevity, then a fairly stern limit must be drawn.

Sections, formulae and exercises are referred to by similar conventions. Thus, "section 5" means section 5 of the current chapter,

while "section (3.5)" means section 5 of chapter 3. "Ex. 3" means Exercise 3 of the current section, "Ex. (4.3)" means Exercise 3 of section 4 in the current chapter. "Ex. (6.4.3)" means Exercise 3 of section (6.4).

I shall be most grateful for any comments which readers may care to offer.

My present debts of gratitude are too numerous to list; I must recall, however, the pleasant and instructive years that I spent at the Applied Mathematics Laboratory of the New Zealand Department of Scientific and Industrial Research.

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Manchester

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CHAPTER 1

INTRODUCTION

1.1 Scientific prediction

A title such as "prediction theory" is apt to raise unjustified expectations, because any claim to foretell the future, in however trivial and limited a sense, can never be regarded as something completely matter of fact. For this reason, we hasten to define our position.

Ideally, prediction is a by-product of the quantitative understanding of a situation, of a *physical model.* Thus, knowledge of Newton's laws of motion enables one to predict the paths of the planets with extreme accuracy; the laws of dynamics and elasticity enable one to predict the motion of a structure such as a bridge or a building under the influence of given applied forces. The majority of situations are *stable to specification*, in that a model which is only approximately valid will yield predictions which are also approximately valid, at least over a limited interval of time. Thus, Newton's laws do not allow for relativistic effects, and the model assumed for an engineering structure will certainly be oversimplified, but the derived predictions will not be greatly in error initially.

A model is termed *stochastic* or *deterministic* according as to whether it does or does not contain random variables. The classic models (such as the two examples above) are deterministic, and give definite predicted values. On the other hand, the future of a stochastic process is only partly determined by past values of the variables, and the idea of a definite prediction must be replaced by that of a *conditional distribution*; a probability distribution of future values, conditioned by the knowledge of past values.

Probabilistic effects may enter a model in a fundamental and inescapable fashion, as in quantum mechanics. Alternatively, they may enter because one is essentially interested in the average or "typical" behaviour of a complex system rather than in its detailed behaviour; this is the situation in statistical mechanics. Finally (and this situation is related to the last one), they may enter because there is a host of minor deterministic effects which one could not treat even if one would; the statistical effects represent this residuum of complicated and unexplained deterministic variation. The "residual" of a statistical model in econometrics, say, represents unexplained (but presumably explicable) variation of this type. (For the sake of brevity we have made these statements in a rather definite and unqualified fashion. In fact, they touch on the basic question of determinism, and are all controversial to some extent.)

1964

SPECTRAL ANALYSIS OF ECONOMIC TIME SERIES

BY C. W. J. GRANGER

IN ASSOCIATION WITH

M. HATANAKA

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PREFACE

A time series is a sequence of data, almost certainly intercorrelated, each of which is associated with a moment of time. As the majority of data in economics is found in the form of time series it has long been recognized that the development of sophisticated and powerful methods for analyzing such series is of importance when questions such as the testing of economic hypotheses are discussed. Many of the earlier methods have not proved satisfactory and it is thus clearly sensible to ask if methods used with success in other disciplines could not be used in economics. The most obvious of the new methods that needed to be considered was spectral analysis.

The main object of this volume is to report upon the methods developed by members of the Time Series Project at the Econometric Research Program of Princeton University. The Project was directed by Professor Oskar Morgenstern and was also particularly fortunate in being advised from its earliest days by Professor John Tukey who made available to us many of his unpublished methods of analysis. The other members of the Econometric Research Program particularly concerned with the Time Series Project were Michael Godfrey, Michio Hatanaka, Herman Karreman, Mitsuo Suzuki, Thomas Wonnacott, and the author.

Upon being initiated into the use of spectral methods by Professor Tukey, we were soon convinced of their potential importance, particularly the use of cross-spectral methods to discover and describe the possibly complex inter-relationships between economic variables. There were, however, two main obstacles to the direct use of these methods to economics. The first was that a re-orientation of many of our previous concepts of economic theory was required before the results of spectral analysis could be usefully interpreted. To non-mathematicians, the usefulness of presenting results in terms of frequencies is not immediately obvious but it is felt that once the great flexibility of this approach is appreciated it becomes important to recast many economic hypothesis in terms of frequencies instead of distributed lags or simple correlations.

Our second main obstacle was that, strictly, spectral methods may only be used on stationary data whereas it is obvious that few if any economic series are stationary. Much of the effort of the Project has been directed to the problems of either how to make economic data less non-stationary or to discovering whether or not spectral methods can extract useful information from series which are not completely stationary. The first of these problems chiefly concerns the development of efficient methods of removing trends in means. We are now confident, using the methods outlined in Chapter 8 and given sufficient data, that any undesirable effects due to such trends can be minimized. The effects of

certain other forms of non-stationarity on spectral methods have been investigated both empirically and, in a non-rigorous manner, theoretically in Chapter 9. Although the study of non-stationary series is in its very early stages, we are at the moment confident that spectral methods can provide useful results when used with non-stationary series generated by processes which appear intuitively realistic for economics.

This book attempts both to promote the use of methods of analysis which are new to economics and to present and justify some entirely new methods in time series analysis. This has presented many problems of presentation as these two diverse objectives require two entirely different levels of mathematical sophistication. Thus an attempt has been made to follow each mathematically-orientated section or chapter by a non-mathematical discussion of the results. These non-mathematical sections often proceed in terms of analogies and intuitive reasoning. It should be remembered, however, that many of the basic concepts used in spectral methods are conceptually difficult, particularly the Cramér representation of a stationary series and the basic implications of the cross-spectrum. It is hoped that the non-mathematical sections will aid comprehension; but readers should be warned that if the analogies and the semi-intuitive reasoning are carried too far they are likely to give misleading results.

The presentation is chiefly aimed at graduate students taking advanced courses in econometrics, econometricians, and those statisticians who advise economists, although it is hoped that a much wider class of research workers will also find new concepts and techniques which will be useful to them.

It appears to be impossible to choose a perfect moment to record the methods of a quickly developing field. At all times, new and exciting possibilities are being considered, new results are appearing, and fresh experience is accumulated. Thus it is felt that any book on spectral methods appearing at the present time must be in many ways incomplete and should be considered as a progress report. Since the first draft was prepared, for example, a number of important and basic ideas in the field of non-stationary series have been proposed, new information is available concerning the problem of removing annual fluctuations from a series and a new method of constructing highly efficient filters has been discovered. If time were taken to insert these developments into the text, doubtless further results would by then become available, and so forth. Nevertheless these new results mainly represent further developments of a number of basic methods and as no full account of these methods is at present available it is hoped that their presentation here at this time is justified.

Although the whole Time Series Project has been a team effort, certain aspects of the work have been particularly connected with certain

people and I feel that these should be acknowledged. The interpretation of cross-spectra and the demodulation technique were shown to us by John Tukey, the partial cross-spectral methods were developed by Thomas Wonnacott, and the considerable and important task of preparing the numerous computer programs for our various computers has been organized and, for the large majority, personally carried out by Herman Karreman. (A research memorandum, No. 59, giving the more important of these computer programs in Fortran language for use on an IBM 7090 computer is available from the Econometric Research Program, Princeton University, 92A Nassau Street, Princeton, N.J., U.S.A.)

I would like to thank Oskar Morgenstern for his continual encouragement and inspiration; Michio Hatanaka for his help at all stages in the preparation of this book and in particular for the two important studies using spectral methods which appear as chapters twelve and thirteen; John Tukey for showing us his extremely effective, if somewhat individualistic, methods of analyzing a time series; both Mitsuo Suzuki and Herman Karreman for advising and correcting the first draft of the manuscript; my wife Patricia for help with numerous details, and Mr. J. Wilson of Princeton University Press for preparing the book for publication. I would also like to thank numerous friends and acquaintances who by constant help and discussion have helped form the climate of opinion and experience within which this book was written. I would finally like to thank Professor Tew and Professor Pitt of the University of Nottingham, England, who have encouraged and enabled me to make the four trips to the United States which has made my part in this work possible.

I would like to emphasize that the views and opinions expressed in the book are mine alone and that any remaining errors are entirely my own responsibility.

The Time Series Project has been supported by a grant from the National Science Foundation and some individual members of the Project have been partially supported by grants from the office of Naval Research and the Rockefeller Foundation. My own first year in Princeton was as a Harkness Fellow of the Commonwealth Fund.

The manuscript was typed by Mrs. Lois Crooks, Mrs. Helen Peek, and Miss Helen Perna to whom I also wish to express my gratitude.

C. W. J. Granger,

Princeton, August, 1963.

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1964

MATHEMATICAL AND STATISTICAL TECHNIQUES FOR INDUSTRY

MONOGRAPH NO. 1

MATHEMATICAL TREND CURVES: AN AID TO FORECASTING

J. V. GREGG, B.A.
C. H. HOSSELL, M.A.
J. T. RICHARDSON, B.Sc.

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OLIVER & BOYD

FOREWORD

This monograph has been written by a small team of people in ICI who have made a close study of mathematical trend curves and their value in long-term forecasting. It is one of a number prepared for the Statistical Methods Panel of ICI, which was set up to collate and disseminate mathematical statistical techniques of value in dealing with various problems, including those met in the techno-commercial and administrative fields. It is intended to publish further monographs on the application of such techniques.

Forecasting is indispensable in commercial and manufacturing activities, and forecasts are essentially subjective judgments made on the basis of existing information. It is usual and prudent to adopt alternative procedures in assessing the future and then to compare their results. There are various ways of basing forecasts on data available for a series of years. The most common is to analyse the economic, technical and commercial factors which have influenced the past figures and then, on the basis of assumptions on how these factors will operate in the future, to build up the forecast. Another method is to graph past data and, by use of a suitable trend curve, to extrapolate the past growth into the future. In view of the use of such curves as an alternative method, it is surprising that relatively little appears to have been published on the types of curves which have found favour and, as far as is known, no attempt has been made objectively to decide what curve or curves are at least consistent with the known data for the past.

The authors have tried to remedy this deficiency by writing a comprehensive document on trend curves and suggesting a method which affords some discrimination when the choice of a trend curve is being considered.

CONTENTS

1966

Time-Series Computations in FORTRAN and FAP

VOLUME I — A PROGPAM LIBRARY

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Preface

In the fall of 1952 I joined, as a graduate student, a Massachusetts Institute of Technology project called the Geophysical Analysis Group, and so began a twelve-year effort in the application of digital computers to time-series problems. This project, the G.A.G., was organized by Professors G.P. Wadsworth and P.M. Hurley of M.I.T. and by Dr. Daniel Silverman of the Stanolind Oil and Gas Company. It assumed the task of attempting the realization of Norbert Wiener's time-series concepts on the Whirlwind I (WWI) Computor in the echo-sounding problems of seismic exploration for oil.

At the same time I developed a close friendship with my fellow student Enders A. Robinson, on whom the directorship of G.A.G. soon devolved. Robinson's efforts centered in the elucidation of theory and its translation to discrete notation, and my own work tended toward machine realization of theory, but we each made sufficient excursions into the other's domain to form a profitable research partnership. This pattern has persisted over the years.

The Geophysical Analysis Group is relevant for the reason that many of the programming concepts presented herein were seeded in the 16-bit registers of WWI for the seismic exploration problem. Digital prediction, both single and multiple, special digital filtering, spectral and correlation analysis, traveling spectral analysis, automatic processing systems for multitrace seismograms, and many other operational concepts were developed and experimented with on WWI to an unprecedented degree. Besides myself and Robinson those involved with computation included Mark Smith, Howard Briscoe, William Walsh, Robert Bowman, Freeman Gilbert, Sven Treitel, Donald Grine, Kazi Haq, Donald Fink, Robert Wylie, Manuel Lopez-Linares, Richard Tooley, and Robert Sax. The ideas carried into industry and pursued there by students associated with G.A.G. have now ripened to the point of causing what amounts to a technological revolution in seismic interpretation.

In 1954 Robinson left, eventually to become Associate Professor of Mathematics at the University of Wisconsin, and I assumed directorship of G.A.G. until its termination in 1957, but frequent visits with each other kept alive our mutual interests.

With G.A.G.'s termination and the subsequent retirement of WWI, I was forced to the realization that my programming output might just as well have been expressed in vanishing ink—an experience which rankled long and which underlies our determination to develop stable programming and communicating techniques.

I took a year's leave of absence from my Assistant Professorship in the Department of Geology and Geophysics at M.I.T. and spent it in military applications of special-design general purpose computers with RCA. This work tended to keep me from recognizing the latent power of the then infant language of FORTRAN.

... I kept my hand in programming on the IBM 704, but it was not until 1960, when I was asked by the Advanced Research Projects Agency to set up a project like G.A.G. but focused on the underground detection problem of VELA UNIFORM, that I became seriously involved with the new computers. I was fortunate in being able to attract Robinson to the project, as well as many gifted graduate students.

By this time FORTRAN had become well established, and, after some hesitancy, we began to use it, gradually evolving a sense of proportion in the mixture of FORTRAN and FAP programming. I find in this mixture that the whole is greater than the sum of its parts. For not only can we have the essential power of the individual languages, but they can supplement each other's weaknesses, as, for instance, they do when we use subroutine sandwiches of alternating language or use FAP programs to bolster FORTRAN's capabilities.

Once again this leaves me committed, albeit partially, to a machine language. But the situation is not as bad as it was ten years ago. In the first place, the ubiquity of the IBM 700 series machines suggests a national and international investment in specific hardware and software of considerable inertia. The time constant of change has lengthened to a point where we should be able to keep up with it without periodic wholesale abandonment of past results. Secondly, our program design, testing, and documentation techniques have matured to the point where machine language translation is not nearly as formidable a prospect as previously.

These considerations, the rapid advances which have been made in time-series computations, the growing requests we have had for the programs, and the general expanding interest in time series and in programming, have all encouraged me to pause and to pull together the myriad threads of our work into a single document representing, in first approximation, where time-series computations stand with respect to today's machines. Such has been my goal. However, this goal has proved too ambitious for a single volume, and we content ourselves in Volume I with a presentation of our subroutine library per se. Volume II will be devoted to the development of pertinent time-series theory from the computational viewpoint, to the consideration of computational applications in a realistic setting, and to discussion of programming technique.

Taken together, the first and second volumes of Time-Series Computations in FORTRAN and FAP may be considered an introduction to a new topic, namely, the realization of modern time-series theory on digital computers. Their principal intended audience is students of time series or communications engineering who wish to acquire advanced techniques of handling empirical time series with present-day computational equipment, especially on the IBM 709, 7090, or 7094. By "advanced" I refer both to the conceptual level of the techniques and to the professionalism of their realization.

But I would hope that this work, Volume I especially, should also prove of value to the general programming community. The majority of our programs are not specialized to the time-series area. What we have done is to fill the gap between basic FORTRAN statements and time-series operations with a complex of general-purpose black boxes that could be used to assist in the development of other areas of application. But even aside from functional utility, we hope that all computing groups faced with the problems of program exchange and communication will be interested in our experiments in communication formalisms.

The subroutine library constitutes the bulk of Volume I. It represents the distillation of years of effort of my co-workers and myself. Cost studies of programming systems of this size (about 40,000 registers) might predict a developmental price tag of about a quarter million dollars for this set. Consequently we have felt justified in devoting considerable time and effort to the development of techniques for communicating our results in the context of applied problems.

At the lowest level of communication, that is, the individual subroutine, we have tried to maintain high standards both of programming and of documentation. Toward the latter end, we have adhered to a program-writing format which might be called the self-documenting symbolic deck. In this format, the program card deck contains a program abstract and a detailed input-output specification, as well as illustrative and critical examples. The card deck is totally definitive of its own behavior.

The format was originally designed for input to an automatic debugging compiler which would read the examples, set up appropriate test programs, execute the test programs, and report back results. In the press of other business the compiler never proceeded beyond a rudimentary stage, but the format has remained and proved itself valuable in our own internal communications.

Furthermore, the format has proved itself many times over as a disciplining device for keeping programmers honest. It is a characteristic of the trade that programmers modify and remodify their decks. The juxtaposition of the documentation and the program proper in deck listings emphasizes documentation errors that result from such modifications, and the weeding out of these errors becomes a natural and integral part of the debugging process. Moreover, to a programmer, there is a great psychological difference between having to change a few comment cards and tracking down a secretary to make the same changes on a mimeograph master in order to run off an updated memorandum.

The self-documenting program deck is a black box with input-output terminals fully described. It is necessarily bulky, the description being generally several times the length of the program proper. For routine reference we turn to compressed summaries, the "program digests," which, by judicious choice of terminology, enable one familiar with the programs to refresh his memory of calling-sequence details needed while programming, with an absolute minimum of page turning. For general scanning of and access to the programs, we have sorted them by various functional and nonfunctional attributes. The other types of documentation in Volume I relate to subroutine library structure and are of more specialized interest to the system programmer.

But the study of n black boxes, each of which performs an isolated task in time-series analysis, does not give one a sense of the coherency of the subject, or of the methods of interconnecting the boxes in broad experimental applications. For such purposes we have designed the experimental programs to be presented in Volume II. Each of these programs represents a series of experimental studies in an interconnected area of time-series analysis, with some carry-over from one program to the next. They permit the reader to see essentially all of our subroutines used in an applied framework.

The applications chosen for illustration in Volume II range from elementary ones to problems the average student or research worker is unlikely to have encountered (especially multi-input processes). Since our theoretical development of time series is of rather limited scope, we have included appendixes on some of the less well-known topics covered in the experiments.

The experiments of Volume II are designed to be readable without knowledge of the basic machine language, FAP, and to require a minimum of experience with FORTRAN. The study of Volume II, especially in conjunction with practice on a computer which can accept the subprograms of Volume I, is probably the easiest way of acquiring familiarity with the techniques we have to offer.

It is an unfortunate fact that artificial but general languages like FORTRAN are, in themselves, incapable of expressing many of the critical time-series operations in truly efficient form. This situation may change, but probably not in the near future. To a large extent our subroutine library may be viewed as an interdependent collection of FORTRAN and FAP programs where the FORTRAN programs steer the FAP programs to the desired task. The higher-level FORTRAN programs will easily compile

on machines outside the IBM 700 series family, but their required subordinates, the FAP workhorse programs, will not in general carry over without hand-coded translation.

For this reason, Volume II will present expositions of the more important logical processes used in the FAP subprograms to attain high-speed behavior, particularly in connection with correlation and spectral analysis. A knowledge of FAP is desirable but is not essential, since we lean considerably on ordinary flow charts for detailed relationships.

Other limitations of a formal nature inherent in FORTRAN II have led us to some programming effort in the twilight region between FORTRAN and FAP, that is, to the writing of FAP programs which utilize "forbidden" knowledge of the FORTRAN system in order to remove these limitations, and which we therefore label "system-expansion programs." Volume II includes a discussion of the techniques and problems involved in such programming and should prove of interest to serious students of programming.

In short, then, we have limited the first volume to the presentation of the subroutine library with subsidiary documentation designed for the working programmer, and we have reserved time-series and programming concepts for Volume II.

The "we" I use frequently above is not editorial, but includes my many co-workers, mostly graduate students, who have contributed to the subprogram collection and with whom it has been my pleasure to work. In this congenial and loosely structured group, considerations of programming technique and style were developed to refined levels. Although the authorship of the programs is given individually, I would like to emphasize the importance of the contributions of James Galbraith, Jon Claerbout, and most particularly Ralph Wiggins. Other students directly associated were William Ross, Cheh Pan, Carl Wunsch, and Roy Greenfield.

As for what theory we include in Volume II, much of it is pure review, but some of it has previously appeared only in project report form. I consider Robinson's solution, in the fall of 1962, of the multi-input iteration problem to be a significant achievement. Wiggins pursued and expanded the analysis from this base through the program-development stage, and in so doing was the first to demonstrate the computational feasibility of multi-input least squares.

But the work presented here has also depended on many others. The tireless and dedicated writing of test routines by Joseph Procito has been invaluable in the establishment of program reliability. In broader areas of service programming, analysis, data handling, desk calculating, etc., we also relied on Mrs. Irene Hawkins, Karl Gentili, my wife Jacqueline, Ervinia Irbin, Mrs. Susan Kannenberg, Allan Kessler, and Lloyd Kannenberg. Most of the card preparation and manuscript chores fell to Mrs. Elizabeth Studer, to my wife, and to Mrs. Wendy Tibbets, with assistance from Mrs. Elene Hershberg, Dauna Trop, Mrs. Myrna Kasser, Regina Lahteine, Mrs. Hazel White, and Mrs. Barbara Cullum.

The punched-card work involved in these two volumes is too elegant to be passed over without further comment. The conventions and forms that we now use regularly (not all of which appear in these volumes; for instance, the mathematics of Volume II was card-coded in the source manuscript) I consider to be significant experiments in a field — call it "punched-card typography" — of growing importance in printing. In large part these conventions are due to my wife, who has become our arbiter of formats and to whose sense of style and standards of excellence we are much indebted.

Over the years we have been favored with the most friendly cooperation of the machine operators and supervisors, starting in the early Whirlwind days with Robert A.J. Gildea (to whom I also owe many enjoyable hours of chess while waiting for the machine to come back) and Michael Solomita, and continuing with Anthony Sacco at the M.I.T. Computation Center and at the Cooperative Computing Laboratory at M.I.T., John Harmon and our long-term friend Michael Saxton of IBM, and more recently with Thomas Burhoe, Mason Fleming, and William Jarvis of IBM.

We owe much to the sponsors of both the G.A.G. project and the VELA UNIFORM project for the computing facilities these projects have afforded us in the development of time-series and computing concepts, and to Lincoln Laboratory, the M.I.T. Computation Center, and Geoscience Incorporated for the use of programs developed under their auspices.

Concerning editorial assistance, I am indebted to Robinson for critical review of the mathematical aspects of the manuscript and to Wiggins for his joint labors with my wife and myself in the editing of the programs.

It is indeed a pleasure for me to acknowledge the many contributions and accommodations from this small army of co-workers and associates.

Brookline, Massachusetts
November, 1965

S.M.S., Jr.

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Time-Series Computations in FORTRAN and FAP

The sorted lists which will follow below need some introduction with regard to format. First of all, the sortings have been made on the basis of names of principal entries, and the lists are alphabetically ordered with respect to these names. In the case of multiple-entry programs, the names of the secondary entries appear as a parenthetical list following each appearance of the principal entry name. However, a parenthetical list following a name is not necessarily a list of secondary entries; it may alternatively be a list of functionally related programs. For example, each appearance of the Fourier-transform program QFURRY is followed by a parenthetical reference to the inverse Fourier-transform program QIFURY, and conversely.

Secondly, it should be noted that we run into an occasional problem resulting from the fact that the present sortings are necessarily based on six-character names for the principal entries, whereas in the program listings of Section 10 we sometimes have appended serial numbers and/or computer numbers to distinguish between programs of identical principal entry names. The sortings have been made on the basis of effective names. The effective names are identical to the principal entry names in cases where no ambiguity can arise. Effective names for the exceptional cases are listed below.

Effective Name	True Name	Effective Name	True Name
CLOCK1	CLOCK1 (7090)	LINE	LINE (709)
CNVLV2	CONVLV -II	LINE90	LINE (7090)
DISPLA	DISPLA (709)	LINEH	LINEH (709)
DSPL 90	DISPLA (7090)	LINH90	LINEH (7090)
FRAME	FRAME (709)	LINEV	LINEV (709)
FRAM90	FRAME (7090)	LINV90	LINEV (7090)
FT24II	FT24 -II	MULK2	MULK -II
HST2	HSTPLT -II	SETK2	SETK -II
HST309	HSTPLT -III (709)	SETKS2	SETKS -II
HST390	HSTPLT -III (7090)	TIMA2B	TIMA2B (7094)

PROGRAMS SORTED BY FUNCTION

```
*******************************
* 1. ADMINISTRATIVE PROGRAMS *
*******************************

FOR CONTROL OF PROGRAM FLOW
     INDEX -(CHUSET,  SETAPT,  SETEST,  VINDEX), SEVRAL  (PLURAL).

FOR EXPANDING SYSTEM CAPABILITY
   FNDFMT,    GETX  ( IGETX), LOCATE  (   ARG,     CALL,    CALL2,  RETURN,
   SETSBV,    SETUP,   STORE,   WHERE,    XARG,  XINDEX,    XNAME,  XNARGS),
   CNLINE  ( (STM),   (STMO),   (STMM)), PLURNS,   RDATA,  AEREAC  (ENCFIL,
   EOFSET,    (TSM),  (TSMM)), RPLFMT,     SAME  ( XSAME), SEVRAL  (PLURAL),
   VARARG.

FOR UNORTHODOX SUBROUTINE USAGE
   LOCATE  (    ARG,    CALL,    CALL2,  RETURN,  SETSBV,   SETUP,    STORE,
    WHERE,    XARG,  XINDEX,    XNAME,  XNARGS), PLURNS,  SEVRAL  (PLURAL),
   VARARG.

FOR INDEX LOGIC
   FASTRK,    GETX  ( IGETX),   INDEX  (CHUSET,  SETAPT,  SETEST,  VINDEX),
   LOCATE  (   ARG,    CALL,    CALL2,  RETURN,  SETSBV,   SETUP,   STORE,
    WHERE,    XARG,  XINDEX,    XNAME,  XNARGS).

FOR DOCUMENTING EXECUTIONS
   DADECK,  LISTNG,  MEMUSE,   RDATA,  XLCOMN.

FOR EQUIPMENT CONTROL
   CARIGE,   CLKON,    FRAME  (FRAM90),  FSKIP,  ONLINE  ( (STM),   (STMO),
   (STMM)),-REREAD  (ENDFIL,  EOFSET,    (TSM),  (TSMM)),  ASKIP,   SETINO,
   SWITCH,  TRMINO,  ZEFBCD  (ZEFBIN).

FOR PROGRAM TIMING
    CLKON,  CLOCK1,  TIMA2B,  TIMSUB  (INTMSB).

FOR ABSOLUTE MEMORY INFORMATION
   IXCARG,     LOC,  MEMUSE,  XLCOMN,    XLOCV.

FOR SUBROUTINE LIBRARY STUDY
  (NO ENTRIES FOR THIS CATEGORY)

FOR PROPER USE OF MISNAMED VARIABLES
     SAME  ( XSAME).

****************************
* 2. INPUT-OUTPUT PROGRAMS *
****************************

FOR BCD INPUT TO CORE
   RDATA,  REREAD  (ENDFIL,  EOFSET,   (TSM),  (TSMM)), ZEFBCD  (ZEFBIN).

FOR BINARY INPUT TO CORE
   INDATA,  PACDAT,  ZEFBCD  (ZEFBIN).
```

PROGRAMS SORTED BY FUNCTION

```
FOR BCD OUTPUT FROM CORE
   COLABL,    CSOUT,  CVSOUT,  DISPLA  (DSPL90), FMTOUT,  ML12A6,    POUT,
   MOUTAI,   ONLINE  ( (STM),   (STMO),  (STMM)), PWMLIV,  VECOUT,    VOUT,
   VRSOUT,    VSOUT.

FOR BINARY OUTPUT FROM CORE
   OUDATA,  WRTDAT.

FOR GRAPHICAL OUTPUT FROM CORE
   CNTRDB,  CNTROW,  CONTUR,  DISPLA  (DSPL90),  GRAPH,  GRAPHX,  MSTPLT
  ( MST2,  MST309,  MST390),    LINE  (LINE90),  LINEH  (LINM90),  LINEV
  (LINV90), PLOTVS,  PLTVS1.

FOR FORMAT PURPOSES
   COLABL,  DSPFMT,  FNDFMT,  RPLFMT.

***************************
* 3. DATA TRANSMISSION    *
*    AND ACCESS PROGRAMS  *
***************************

FOR STORAGE-TO-STORAGE MOVEMENT
   EXCHVS,     MOVE,  MOVECS,  MOVREV,    MRVRS,  MVBLOK.

FOR STORAGE-TO-TAPE MOVEMENT
   GETRDI,  OUDATA,  WRTDAT.

FOR TAPE-TO-STORAGE MOVEMENT
   INDATA,  PACDAT.

FOR TAPE-TO-TAPE MOVEMENT
   CPYFL2,  DADECK.

FOR INFORMATION STORAGE
   OUDATA,     PAKN  (UNPAKN), SETINO,  TRMINO,  WRTDAT.

FOR INFORMATION RETRIEVAL
     GETX  ( IGETX), INDATA,  LISTNG,     NTHA  ( XNTHA), PACDAT,  UNPAKN
  (  PAKN).

**********************************
* 4. DATA FORM-CHANGING PROGRAMS *
**********************************

FOR CONVERTING DATA MODE
     FIXV  ( FIXVR), FLOATM,  FLOATV,  FXDATA  (FLDATA), MVTOIV  (IVTOMV),
   INTMOL,  ITOPLI,  IVTOMV  (MVTOIV), ML12A6,   XFIXM.

FOR PACKING DATA
   PAKN  (UNPAKN).

FOR UNPACKING DATA
   UNPAKN  (  PAKN).

FOR SCALING DATA
   FXDATA  (FLDATA), MLISCL,  SCPSCL,  SMPSON.
```

PROGRAMS SORTED BY FUNCTION

```
FOR NORMALIZING DATA
    FXDATA  (FLDATA), NMZNGL,  NRMVEC.

FOR ROUNDING DATA
     FIXV  ( FIXVR), FXDATA  (FLDATA),   RND  ( RNDDN,   RNDUP),   RNCV
  (RNDVDN,  RNDVUP),   XDIV  ( XDIVR), XDVIDE  (XDVIDR),  XFIXP,  XVDVBV
  (XDVRBV).

FOR SHIFTING DATA
    MLADJ  ( MRADJ), ITOML,   LSHFT  (XLSHFT), SHFTR1,  SHFTR2.

FOR CHANGING DATA SPACING
    MOVREV.
```

```
******************************
* 5. DATA GENERATING PROGRAMS *
******************************
```

```
FOR GENERATING HOLLERITH
    GENHOL,  GETHOL,  GNHOL2.

FOR GENERATING RANDOM NUMBERS
    GETRDL.

FOR GENERATING SINUSOIDS
    COSTBL  (COSTBX,  SINTBL,  SINTBX), SEQSAC  (NEXCOS,  NEXSIN).

FOR GENERATING SCALARS
     SETK  ( SETKS,  SETVEC),  SETK2,   SETKP  (SETVCP), SETKS2.

FOR GENERATING 1-D ARRAYS
     SETK  ( SETKS,  SETVEC),  SETKP  (SETVCP),  SETKV,  SETKVS,  SETLIN
  (XSTLIN), SETLNS,    STZ,    STZS.
```

```
**************************
* 6. DATA INQUIRY PROGRAMS *
**************************
```

```
FOR FINDING EXTREMAL VALUES
    MAXSN  ( MAXAB,   MINAB,   MINSN), MAXSNM  (MAXABM,  MINABP,  MINSNM).

FOR COMPARING DATA
    CMPARP  (CMPARS), CMPARV  (CMPARL),  CMPRA  (CMPRFL,  XCMPRA),   INDEX
  (CMUSFT,  SETAPT,  SETEST,  VINDEX), LIMITS,  LOCATE  (   ARG,    CALL,
    CALL2,  RETURN,  SETSBV,   SETUP,   STORE,   WHERE,   XARG,  XINDEX,
    XNAME,  XNARGS), XACTEQ,  XLIMIT.

FOR SEARCHING DATA
    FASCNL,  FASTRK,  NXALRM,  SEARCH,   SRCHL.

FOR SELECTING DATA
    CHOOSE,    GETX  ( IGETX),   NTHA  ( XNTHA),  WHICH  (XWHICH).

FOR ORDERING DATA
    SIZEUP  (SIZUPL).

FOR CLASSIFYING DATA
    MONOCR,   XOOZE.
```

PROGRAMS SORTED BY FUNCTION

```
*********************************
* 7. ELEMENTARY NUMERICAL PROGRAMS *
*********************************
```

```
FOR ADDITION
     ADDK  ( ADDKS,    DIVK,   DIVKS,    MULK,   MULK2,    SUBK,   SUBKS,
    XADDK,  XADDKS,   XDIVK,  XDIVKS,   XDVRK,  XDVRKS,   XMULK,  XMULKS,
    XSUBK,  XSUBKS),  BOOST  (DPRESS,  XBOOST,  XDPRSS), FAPSUP,  NRMVEC,
      SUM  (  XSUM), VPLUSV  (VMNUSV,  XVMNSV,  XVPLSV).

FOR SUBTRACTION
     ADDK  ( ADDKS,    DIVK,   DIVKS,    MULK,   MULK2,    SUBK,   SUBKS,
    XADDK,  XADDKS,   XDIVK,  XDIVKS,   XDVRK,  XDVRKS,   XMULK,  XMULKS,
    XSUBK,  XSUBKS),  BOOST  (DPRESS,  XBOOST,  XDPRSS),  REMAV,  VPLUSV
  (VMNUSV,  XVMNSV,  XVPLSV), XREMAV.

FOR MULTIPLICATION
     ADDK  ( ADDKS,    DIVK,   DIVKS,    MULK,   MULK2,    SUBK,   SUBKS,
    XADDK,  XADDKS,   XDIVK,  XDIVKS,   XDVRK,  XDVRKS,   XMULK,  XMULKS,
    XSUBK,  XSUBKS),  DUBLX  ( DUBLL,   HALVL,   HALVX), MLTSCL,   MULK2,
   MULPLY,  VTIMSV  (XVTMSV).

FOR DIVISION
     ADDK  ( ADDKS,    DIVK,   DIVKS,    MULK,   MULK2,    SUBK,   SUBKS,
    XADDK,  XADDKS,   XDIVK,  XDIVKS,   XDVRK,  XDVRKS,   XMULK,  XMULKS,
    XSUBK,  XSUBKS), DIVIDE,   DUBLX  ( DUBLL,   HALVL,   HALVX), VDVBYV,
     XDIV  ( XDIVR), XDVIDE  (XDVIDR), XVDVBV  (XDVRBV).

FOR MODIFYING SIGN
    ABSVAL,  CHPRTS  (RVPRTS), CHSIGN,  MOVREV.

FOR RAISING TO POWERS
    MVSQAV,   POWER  (SMPRDV), SQRMLT,  SQUARE  (XSQUAR).

FOR TAKING ROOTS
    SQROOT,  XSQRUT.

FOR TRIGONOMETRIC FUNCTIONS
    ARCTAN,  SEQSAC  (NEXCOS,  NEXSIN).

FOR COLLAPSING VECTORS
    COLAPS,  KOLAPS.

FOR ROTATING VECTORS
    ROTATL.

FOR REVERSING VECTORS
    CHPRTS  (RVPRTS), MOVREV,   REVER,  REVERS.

FOR EXCHANGING VECTORS
    EXCHVS.

FOR REFLECTING VECTORS
    REFLEC  (XRFLEC).
```

PROGRAMS SORTED BY FUNCTION

* 8. MISCELLANEOUS NUMERICAL PROGRAMS *

FOR INTERPOLATION
ARBCOL, EXPAND, INTOPR, LINTRI, QINTRI.

FOR SAMPLE BASE CHANGING
EXPAND, NURINC, SIFT.

FOR GENERATING SINUSOIDS
COSTBL (COSTBX, SINTBL, SINTBX), SEQSAC (NEXCOS, NEXSIN).

FOR TRIGONOMETRIC FUNCTIONS
ARCTAN, SEQSAC (NEXCOS, NEXSIN).

FOR TREATING ODD AND EVEN PARTS
CHPRTS (RVPRTS), SPLIT (REFIT).

FOR FITTING EQUATIONS TO DATA
CUFIT1, INTOPR, LSLINE, PRBFIT, QUFIT1.

FOR CONTOURING
CNTROB, CONTUR.

FOR DELTA AND STEP FUNCTIONS
DELTA (STEPC, STEPL, STEPR, XDELTA, XSTEPC, XSTEPL, XSTEPR).

FOR CONVERTING COMPLEX NUMBERS
AMPHZ (REIM).

FOR MOVING SUMMATION
BLKSUM, MUVADD, PVINAV, MVNSUM, MVSQAV.

FOR INVERTING FUNCTIONS
IFNCTN.

FOR DOT PRODUCTS
DOTJ, FDOT (FDOTR), VDOTV.

* 9. PROBABILITY AND STATISTICS *

FOR FINDING MOMENTS
POWER (SMPROV).

FOR FINDING AVERAGES
AVRAGE, MVINAV, MVSQAV, REMAV, TAMVL (TAMVR), XAVRGE (XAVRGR), XREMAV.

FOR FINDING R.M.S. VALUES
RMSDEV (RMSDAV).

FOR FINDING SUMS OF SQUARES
SQRDFR (SQRDEV), SQRSUM (XSQSUM), XSQDFR (XSQDEV).

PROGRAMS SORTED BY FUNCTION

FOR FINDING SUMS OF DIFFERENCES
SQRDFR (SQRDEV), SUMDFR (SUMDEV, XSMDEV, XSMDFR), XSQDFR (XSQDEV).

FOR GENERATING RANDOM NUMBERS
GETRO1.

FOR RANDOMIZING DATA
SHUFFL.

FOR FINDING DISTRIBUTIONS
FROCT1, FROCT2, POKCT1, PRBFIT, PROB2.

FOR PROBABILITY TRANSFORMATION
GRUP2, MPSEQ1, NOINT1 (NOINT2).

FOR CHI-SQUARE ANALYSIS
CHISQR, KIINT1.

FOR DEPENDENCY TESTING
MSCON1, POKCT1.

FOR NORMAL CURVE INTEGRATION
NOINT1 (NOINT2).

* 10. INTEGRATION AND DIFFERENTIATION PROGRAMS *

FOR DEFINITE INTEGRATION
MVNTIN (MVNTNA), SMPSON, TINGL (TINGLA).

FOR INDEFINITE INTEGRATION
IDERIV (DERIVA), INTGRA (IINTGR), TAMVL (TAMVR).

FOR DIFFERENTIATION
DERIVA (IDERIV), IINTGR (INTGRA).

FOR INDEFINITE SUMMATION
INTSUM (DIFPRS, XNTSUM).

FOR DIFFERENCING
DIFPRS (INTSUM, XDFPRS).

* 11. 2-D ARRAY AND 3-D ARRAY PROGRAMS *

FOR MATRIX MULTIPLICATION
MATML1, MATML3.

FOR MATRIX INVERSION
MATINV, SIMEQ (DETRM).

FOR SOLVING MATRIX EQUATIONS
LSSS1, RLSPR, RLSSR, SIMEQ (DETRM), WLLSFP.

PROGRAMS SORTED BY FUNCTION

FOR DETERMINANT EVALUATION
SIMEQ (DETRM).

FOR MATRIX TRANSPOSITION
MATRA, MATRA1.

FOR MATRIX FACTORIZATION
MFACT.

FOR 2-D ARRAY ROTATION
ROAR2.

FOR INTERPOLATING 2-D ARRAY COLUMNS
ARBCOL.

FOR 2-D ARRAY DOT PRODUCTS
DOTP.

FOR 2-D ARRAY CORRELATIONS
SPCOR2.

FOR 2-D ARRAY FOURIER TRANSFORMATION
PLANSP.

FOR SOLVING 2-D ARRAY EQUATIONS
FIRE2, RLSPR2.

FOR MATRIX VECTOR REVERSAL
MRVRS.

FOR MATRIX VECTOR DOT PRODUCT
MDOT, MDOT3.

FOR MATRIX VECTOR CORRELATION
CRSVM.

FOR SOLVING MATRIX VECTOR EQUATIONS
MIFLS, MIPLS, MISS.

* 12. POLYNOMIAL PROGRAMS *

FOR POLYNOMIAL EVALUATION
FASCUB, IPLYEV, POLYEV.

FOR FINDING POLYNOMIAL ROOTS
MULLER.

FOR POLYNOMIAL MULTIPLICATION
CONVLV, CNVLV2.

FOR POLYNOMIAL DIVISION
POLYDV.

PROGRAMS SORTED BY FUNCTION

FOR POLYNOMIAL SQUARE ROOTS
PSQRT.

FOR SYNTHESIZING POLYNOMIALS
PLYSYN, POLYSN.

* 13. CORRELATIONS AND CONVOLUTIONS *

FOR AUTOCORRELATION
CROSS, CROST, PROCOR (FASCOR, FASCR1, FASEPC, FASEP1), QACORR, QXCORR, WAC.

FOR CROSS-CORRELATION
CROSS, CROST, PROCOR (FASCOR, FASCR1, FASEPC, FASEP1), QXCORR, QXCOR1.

FOR CONVOLUTION
CONVLV, CNVLV2, QCNVLV.

FOR DOT PRODUCTS
DOTJ, FDOT (FDOTR), VDOTV.

* 14. HARMONIC TRANSFORMS *

FOR COSINE TRANSFORMATION
ASPECT, ASPEC2, COSIS1, COSP (COSISP, SISP).

FOR SINE TRANSFORMATION
COSIS1, COSP (COSISP, SISP).

FOR FOURIER TRANSFORMATION
COSIS1, COSP (COSISP, SISP), FT24 (FT24II), QFURRY (QIFURY), XSPECT.

FOR INVERSE FOURIER TRANSFORMATION
QIFURY (QFURRY).

* 15. MISCELLANEOUS SPECTRAL *
* ANALYSIS PROGRAMS *

FOR DANIELL WEIGHTING
ADANL (ADANX, XDANL, XDANX).

FOR SPECTRAL FACTORIZATION
FACTOR.

FOR GENERATING NUMERICAL FILTERS
GNFLT1.

PROGRAMS SORTED BY FUNCTION

FOR CONVERTING TO AMPLITUDE AND PHASE
AMPHZ (REIM).

FOR CONVERTING TO REAL AND IMAGINARY
AMPHZ (REIM).

FOR SPECTRAL COMPARISONS
MXRARE.

FOR GENERATING SINUSOIDS
COSTBL (COSTBX, SINTBL, SINTBX), SEQSAC (NEXCOS, NEXSIN).

1967

STOCHASTIC APPROXIMATION AND NONLINEAR REGRESSION

ARTHUR E. ALBERT
LELAND A. GARDNER, JR.

RESEARCH MONOGRAPH NO. 42
THE M.I.T. PRESS, CAMBRIDGE, MASSACHUSETTS

Preface

This monograph addresses the problem of "real-time" curve fitting in the presence of noise, from the computational and statistical viewpoints. Specifically, we examine the problem of nonlinear regression where observations $\{Y_n : n = 1, 2, \cdots\}$ are made on a time series whose mean-value function $\{F_n(\boldsymbol{\theta})\}$ is known except for a finite number of parameters $(\theta_1, \theta_2, \cdots, \theta_p) = \boldsymbol{\theta}'$. We want to estimate this parameter. In contrast to the traditional formulation, we imagine the data arriving in temporal succession. We require that the estimation be carried out in real time so that, at each instant, the parameter estimate fully reflects all of the currently available data.

The conventional methods of least-squares and maximum-likelihood estimation, although computationally feasible in cases where a single estimate is to be computed after the data have been accumulated, are inapplicable in such a situation. The systems of normal equations that must be solved in order to produce these estimators are generally so complex that it is impractical to try to solve them again and again as each new datum arrives (especially if the rate of data collection is high). Consequently, we are led to consider estimators of the "differential correction" type. Such estimators are defined recursively. The $(n+1)$st estimate (based on the first n observations) is defined in terms of the nth by an equation of the form

$$\mathbf{t}_{n+1} = \mathbf{t}_n + \mathbf{a}_n[Y_n - F_n(\mathbf{t}_n)] \qquad (\mathbf{t}_1 \text{ arbitrary}; n = 1, 2, \cdots),$$

where $\{\mathbf{a}_n\}$ is a suitably chosen sequence of "smoothing" vectors. The

term "differential correction" refers to the proportionality of the difference between $\mathbf{t}_{n+1}$ and $\mathbf{t}_n$ (the correction) to the difference between the nth observation, Y_n, and the value that would be predicted by the regression function if $\mathbf{t}_n$ were in fact the "true" parameter value.

The choice of smoothing vectors critically affects the computational simplicity and statistical properties of such recursive estimates. The main purpose of this monograph is to relate the large-sample statistical behavior of said estimates (consistency, rate of convergence, large-sample distribution theory, asymptotic efficiency) to the properties of the regression function and the choice of smoothing vectors. A wide class of smoothing vectors is examined. Some are deterministic and some depend on (are functions of) the observations.

The techniques used in the analysis are, for the most part, elementary and, by now, standard to those who are familiar with the literature of stochastic approximation. However, for the sake of the nonspecialist, we have tried to keep our treatment self-contained. In all cases, we seek the asymptotic properties (large n) of the solution to the nonlinear difference equation which relates $\mathbf{t}_{n+1}$ to $\mathbf{t}_n$.

As a fortuitous by-product, the results of this monograph also serve to extend and complement many of the results in the stochastic-approximation literature.

The structure of the monograph is as follows. Part I deals with the special case of a scalar parameter. Here we discuss probability-one and mean-square convergence and asymptotic distribution theory of the estimators for various choices of the smoothing sequence $\{a_n\}$. Part II deals with the probability-one and mean-square convergence of the estimators in the vector case for various choices of smoothing vectors $\{\mathbf{a}_n\}$. Examples are liberally sprinkled throughout the book. In fact, an entire chapter is devoted to the discussion of examples at varying levels of generality.

The book is written at the first-year graduate level, although this level of maturity is not required uniformly. Certainly the reader should understand the concept of a limit both in the deterministic and probabilistic senses. This much will assure a comfortable journey through Chapters 2 and 3. Chapters 4 and 5 require acquaintance with the Central Limit Theorem. Familiarity with the standard techniques of large-sample theory will also prove useful but is not essential. Chapters 6 and 7 are couched in the language of matrix algebra, but none of the "classical" results used are deep. The reader who appreciates the elementary properties of eigenvalues, eigenvectors, and matrix norms will feel at home.

The authors wish to express their gratitude to Nyles Barnert, who collaborated in the proofs of Theorems 6.1 through 6.3; to Sue M. McKay, Ruth Johnson, and Valerie Ondrejka, who shared the chore of typing the original manuscript; to the ARCON Corporation, the M.I.T. Lincoln Laboratory, the Office of Naval Research, and the U.S. Air Force Systems Command, who contributed to the authors' support during the writing of the monograph; and, finally, to the editorial staff of the *Annals of Mathematical Statistics*, who were principally responsible for the writing of this monograph.

ARTHUR E. ALBERT
LELAND A. GARDNER, JR.

Cambridge, Massachusetts
October 1966

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1967

SPECTRAL ANALYSIS of TIME SERIES

Proceedings of an Advanced Seminar
Conducted by the Mathematics Research Center,
United States Army and The Statistics Department at the
University of Wisconsin, Madison
October 3-5, 1966

Edited by
Bernard Harris

JOHN WILEY & SONS, INC.
New York · London · Sydney

PREFACE

Spectral analysis has become a significant tool in the statistical analysis of stationary time series since the late 1940's, when John W. Tukey in the United States and M. S. Bartlett in England proposed this technique in essentially its present form. During the 1950's, E. Parzen, S. K. Zaremba, Z. A. Lomnicki, U. Grenander, and M. Rosenblatt, among others studied this technique in great detail, considerably extending the frontiers of knowledge in this area. Technical advances in this area have been continuing at a rapid rate ever since. In view of the extensive recent advances in this area, the Mathematics Research Center, U. S. Army felt that this was an area which was eminently suitable for an advanced seminar and this volume is the result of that effort.

Therefore, the Mathematics Research Center, U. S. Army in conjunction with the Statistics Department of the University of Wisconsin held an Advanced Seminar on the Spectral Analysis of Time Series on October 3-5, 1966. Ten papers were presented at the Advanced Seminar, and they are published here in their entirety. This volume contains an additional paper, a general introduction to the subject, prepared by the editor.

The purpose of the advanced seminar was to present a survey of the basic theory of spectral analysis of time series together with an account of some of the more recent developments of significance. Thus, a substantial number of the papers are devoted to some topics of more recent interest, such as polyspectra, coherence, and cross-spectral analysis. The paper by G. E. P. Box, G. M. Jenkins, and D. W. Bacon in this volume points out some alternatives to spectral analysis which are appropriate to statistical inference in time series. The various sessions were chaired by M. E. Muller, D. Slepian, D. L. Hanson, T. W. Anderson, and D. G. Watts.

The editor would like to express his thanks to H. F. Karreman, M. E. Muller, D. G. Watts and H. J. Wertz who assisted in the planning of the program. Professor S. K. Zaremba deserves special

mention for providing substantial assistance and advice during the editing of this volume. Mrs. Gladys Moran deserves a special note of appreciation for serving as secretary of the program committee and taking care of the many problems concerned with the physical arrangements for the advanced seminar. Mrs. Grace Krewson is to be particularly commended for her painstaking efforts in the typing of the manuscripts and preparation of the figures for publication.

Bernard Harris
Madison, Wisconsin
December 19, 1966

CONTENTS

1967

Cybernetics and Forecasting Techniques

by

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Translated by *Scripta Technica, Inc.*

Translation Editor: ROBERT N. McDONOUGH

AMERICAN ELSEVIER PUBLISHING COMPANY, INC.
NEW YORK 1967

PREFACE

Forecasting programs designed for large general-purpose computers constitute an important new tool in the control of production and economics. An example of such "big" forecasting programming is the work of Professor Richard Stone of Cambridge, who computorized the economics of the United Kingdom for 1970.

Nevertheless, small forecasting filters have their own domain of application. They can be realized not only as programs for general-purpose computers, but also as simple analog devices with quick response. The first of such devices was constructed on the basis of the operator of Academician Kolmogoroff's formula by Professor Dennis Gabor at Imperial College (London) in 1955. Since then many other forecasting filters have been designed for different purposes and in accordance with different formulas (algorithms)—for instance, at Kiev Polytechnic Institute, where the authors work.

These different forecasting algorithms are considered, and many new recommendations are given in this book.

The authors discuss three principal methods of forecasting in addition to some others.

1. Forecasting of deterministic processes, i.e., extrapolation and interpolation.

2. Forecasting of stochastic processes, based on statistical forecasting theory.

3. Forecasting based on adaptation or learning of the forecasting filters.

Professor Gabor's filter was a self-learning one. It is shown in the book that the perceptron—the best known cognitive system—can also be used as a simple forecasting filter. Thus, there is no dividing line between cognitive systems and forecasting filters, for forecasting is the cognition of the future. The theory of cognitive systems can be applied to the designing of forecasting filters and, vice versa, the well developed theory

of statistical forecasting can be used in cognitive system design.

The main problem is realization of optimum forecasting precision, the comparison of the precision and simplicity of various algorithms of forecasting. Sometimes, as in the case of control, quick response of the forecasting filters is also important. Some recommendations are given on the basis of a study of the precision of forecasting in the general form; some, on the basis of calculation of examples. All calculations were performed on digital computers.

The examples are taken from the chemical industry, biology, ocean turbulence processes, forecasting of the relief of the Dnieper river bottom, and so forth.

The most important is the original proposal to combine the forecasting method developed for nonstationary processes (presented by Professor Farmer at the second IFAC Congress) with Kolmogoroff's basic method, developed for stationary processes only. The combined method of forecasting yielded good results in forecasting intracranial pressure in neurosurgery.

A special part of the book is devoted to the use of forecasting filters or cognitive systems in production control. Extremum control of the plant should be effected by a combination of open loop control and a corrector, smoothly correcting the characteristics of the open loop part. Cognitive systems and forecasting filters can be used as correctors.

Forecasting filters furnish the only possibility of constructing a control system for periodic processes, since prediction of the result of the process is essential for its control. This problem is also discussed.

CONTENTS

1967

COMPUTER APPLICATIONS IN THE EARTH SCIENCES:

COLLOQUIUM ON TIME-SERIES ANALYSIS

Edited by

DANIEL F. MERRIAM

1967

CONTENTS

1967

Time Series Analysis Papers

EMANUEL PARZEN
Stanford University

Holden-Day

SAN FRANCISCO, CAMBRIDGE, LONDON, AMSTERDAM

CONTENTS

theorem for discrete parameter stochastic processes which are linear. In this paper a central limit theorem is proved for a class of stochastic processes called multilinear by the author.

Ann. Math. Statist. **29** (1958), 299-301.

Math. Review: The main result is as follows. Let $x(t)$ be a strictly stationary stochastic process and introduce for given time points $t_1, t_2, \ldots, t_k$ the characteristic function $\varphi(u_1, u_2, \ldots, u_k)$ of the stochastic variables $x(t_1), x(t_2), \ldots, x(t_k)$ and another characteristic function $\varphi(u_1, u_2, \ldots, u_k; \tau)$ of the increments $x(t_1) - x(t_1 + \tau)$, $x(t_2) - x(t_2 + \tau), \ldots, x(t_k) - x(t_k + \tau)$. Then it is necessary and sufficient for $x(t)$ to be ergodic that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\tau=0}^{n} \varphi(u_1, u_2, \ldots, u_k; \tau) = |\varphi(u_1, u_2, \ldots, u_k)|^2$$

for every choice of $k, t_1, t_2, \ldots, t_k, u_1, u_2, \ldots, u_k$. This is a condition for asymptotic independence in a certain sense.

Ann. Math. Statist. **28** (1957), 329-348.

Math. Review: The author studies the estimation of the spectral density, distribution function or other spectral averages for stationary stochastic processes. The process is supposed to be normal or of a more general form, whose fourth order mixed moments do not differ too much in a certain sense from those of a normal process. Both discrete and continuous processes are treated. After studying some statistical properties of the sample covariances, the author considers the estimation of the spectral density. His main results describe the bias and covariances of certain estimates in asymptotic terms; relating them to the infinitesimal properties of the spectrum and the estimate used. Alternative estimates are discussed from this point of view. It is shown that the whole graph of the estimate can be obtained by an interpolation procedure.

Ann. Math. Statist. **28** (1957), 921-932.

Math. Review: Let $x(t)$ be a continuous parameter stationary process with spectral density $f(\omega)$. The author studies estimates of $f(\omega)$ of the type

$$f_T^*(\omega) = (2\pi)^{-1} \int_{-T}^{T} e^{-iv\omega} k(B_T v)\, R_T(v)\, dv\,,$$

where $k(u)$ is called the covariance averaging kernel, B_T tends to zero as T tends to infinity and $R_T(v)$ is the empirical covariance function. In terms of these quantities one can write down expressions for the asymptotic variance and bias of $f_T^*(\omega)$, valid under certain regularity assumptions.

To simplify the problem the author assumes that it is practically legitimate to deal with these expressions as if T were finite. This makes it possible to study the large sample mean square error as a function of T, B_T, functional form of $k(u)$, etc. This is done in some detail for certain families of $k(u)$ that contain most of the estimates suggested before. In a numerical investigation a very interesting conclusion is reached: the functional form of $k(u)$ is not as important as may have been thought at first glance. The practical consequences of this are discussed; to the reviewer it seems to indicate that the bandwidth is the most important parameter to consider when choosing spectral estimates.

J. Roy. Statist. Soc. Ser. B. **20** (1958), 303-322.

Math. Review: The article for the most part is a discussion of results obtained by the author [see *Stanford University Appl. Math. Statist. Lab. Tech. Rep. No. 36*]. A variety of spectral estimates in the case of stationary time series are considered. Under rather detailed assumptions on the asymptotic behavior of the covariance function $R(v)$ of the stochastic process such as $|R(v)| \leq R_0 e^{-\rho|v|}$, $\rho > 0$, or $\lim_{v\to\infty} e^{r|v|}|R(v)| = R_r$, $r > 0$, estimates are produced which have asymptotically best behavior in the sense of some measure-like mean square error. The character of the "optimal" estimates depends on the rate of decay of the covariance function. One might feel that such an assumption on the rate of decay would not be verifiable unless there was a refined theory in the context at hand specifying the rate of decay.

Technometrics **3** (1961), 133-166.

This is an expository paper on statistical spectral analysis intended for engineers, geophysicists, etc., who want to learn about

major role in our work, namely, the representation of a stochastic process with finite second moments by means of a reproducing kernel Hilbert space.

Ann. Math. Statist. **32** (1961), 951–989.

Math. Review: In this useful survey paper the author discusses some recent developments in the theory of second-order stochastic processes and in time series analysis. It is argued that, in this context, reproducing kernel Hilbert spaces can be used successfully to obtain a unified theory. Taking as a starting point the idea of representing second-order stochastic variables in a Hilbert space, the author describes how this approach can be used in prediction and estimation problems. In this way he obtains the classical representation theorems for stochastic processes as special cases. This is done explicitly for autoregressive processes with discrete or continuous time parameter. Regression analysis (of time series, with known covariances) takes a particularly simple and attractive form: known results are rederived in a simple way and new results are discovered. The author gives a (simultaneous) confidence band for the mean-value function when this is given as a finite linear combination of known functions. This can be used to test hypotheses concerning the mean-value function. Some other problems are discussed: the concept of a density functional when two hypothetical distributions are compared for the stochastic process in question. Also in this context it is convenient to base the discussion on reproducing kernel spaces. The paper concludes with a section on correlation analysis of stationary processes with known mean-value function, and asymptotic results are given for the relevant estimates.

Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. I, Univ. California Press, Berkeley, Calif., 1961, pp. 469–489.

Math. Review: This paper is based on the observation that the analysis of second-order stochastic processes can be simplified by using a reproducing kernel theory. This idea, which seems to be due to Loève, is carried out in detail and results in a unified theory. In particular, the author shows how to arrive at a useful representation of the process in terms of a reproducing kernel. This is exemplified in a later section. It is shown how the regression analysis of the process can be expressed conveniently in this terminology. It is interesting to note the simultaneous confidence band for the regression function given by the author. Such a band should be of considerable usefulness in time series analysis.

Mathematical Optimization Techniques, Univ. of California Press, Berkeley, Calif., 1963, pp. 75–108.

Math. Review: This paper describes a new approach in which a wide class of smoothing and predicting problems is incorporated. The approach is based on the author's idea that reproducing-kernel Hilbert spaces provide a unified framework for such problems, and the theoretical elaborations of this idea were first developed by the author in "Statistical inference of time series by Hilbert space methods, I" [*Dept. of Statist., Stanford Univ. Tech. Rep. No. 23* (1959)]. The present paper is a further development of examples and applications, and problems of prediction, smoothing, smoothing and prediction, parameter estimation, and signal extraction and detection are treated. It is shown that each type of problem has a characteristic statistical structure, which calls for a coordinate system in which there is a natural way of expressing quantities such as inner products and data-handling procedures. One of the important innovations is the treatment of minimum-variance linear unbiased prediction.

Proc. Sympos. Time Series Analysis (Brown Univ., 1962), Wiley, New York, 1963, pp. 155–169.

Math. Review: Further development of a general approach to statistical problems of extraction, detection and prediction of a signal $S(t)$ in the presence of noise $N(t)$. The conditions for absolute continuity of the distribution of $S(t) + N(t)$ with respect to that of $N(t)$ and respective densities are expressed in terms of reproducing kernel Hilbert spaces. If $S(t)$ is a sure function, $P_{S+N} < P_N$ if and only if $S(t)$ belongs to the reproducing kernel space $H(K, T)$ with the kernel $K(s, t) = E[N(s)N(t)]$, $t, s \in T$. Then $dP_{S+N}/dP_N = \exp[(X, S) - \frac{1}{2}(S, S)]$, where $(\cdot,\cdot)$ refers to $H(K, T)$. If $S(t)$ is stochastic, then $R(s, t) = E[S(t)S(s)]$ has to belong to the reproducing kernel space with the kernel $K(s_1, s)[K(t_1, t) + R(t_1, t)]$. The expressions like (X, S) where X is the sample function, are defined as limits (in the mean, almost surely) and not for individual trajectories. The paper overlaps with a paper developed independently by J. Hajek [*Czechoslovak Math. J.* 12 (87) (1962), 404–444; MR 27 #2070].

J. SIAM Control Ser. A 1 (1962), 35–62.

Summary: In sections 1 and 2, it is shown how one may define and obtain a formula for the probability density functional of a normal time series. This formula is used to study the structure of optimum estimators (section 3) and detectors (section 4) by expressing them in a coordinate-free way in terms of inner products in a reproducing kernel Hilbert space. Various ways of evaluating such inner products are discussed in sections 5 and 6. In section 7, it is shown how reproducing kernel Hilbert spaces provide a solution to the problems of minimum mean-square-error linear and non-linear prediction.

Ann. Math. Statist. 33 (1962), 1065–1076.

Math. Review: Let $X_1, X_2, \ldots, X_n$ be independent random variables with common density f. The author examines estimates of the form

$$f_n(x) = \frac{1}{nh(n)} \sum_{j=1}^{n} K\left(\frac{x - X_j}{h(n)}\right)$$

for $f(x)$. Sufficient conditions on h and K ensuring consistency and uniform consistency of f_n are found, as well as conditions ensuring $E[f_n(x) - f(x)]^2 \to 0$. $E[f_n(x) - f(x)]^2$ is evaluated for certain cases in terms of f. From this the author derives the h minimizing $E[f_n(x) - f(x)]^2$ as a function of K and f, and order of magnitude estimates of $E[f_n(x) - f(x)]^2$. Sufficient conditions for the asymptotic normality of $f_n(x)$ and the sample mode θ_n (suitably normalized) are obtained.

North Atlantic Treaty Organization Advisory Group for Aeronautical Research and Development. Report 245. Presented at the Ninth Meeting of the Structures and Materials Panel, held 20–25th April, 1959, in Paris, France.

Summary: This report represents an attempt, based on probability theory, to survey some of the problems involved in evaluating structural safety. Part I is a review of the probabilistic considerations involved in evaluating the strength of materials, and the construction of so called *S-N* curves. In Part II is briefly advanced a probabilistic model for the life before fatigue failure of a structure.

J. Res. Nat. Bur. Standards Sect. D 68D (1964), 937–951.

Summary: This paper attempts to develop a philosophy for empirical time series analysis, involving the routine use of four data handling procedures (covariance estimation, spectral estimation, autoregressive model fitting and spectral estimation, and trend elimination and estimation) embodied in a computer program. The cross-spectral analysis of a pair of time series, each consisting of 4000 observations, requires approximately 10 minutes of [an IBM] 7090, including computation of covariances. Several examples of empirical time series analysis are given.

1967

Statistical communication and detection

with special reference to

digital data processing of radar and seismic signals

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1967
HAFNER PUBLISHING COMPANY
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PREFACE

This monograph treats several aspects of statistical communication and detection theory with special reference to the digital data processing of radar signals and seismic signals in the presence of noise. By bringing together the two different disciplines of radar and seismology under one cover, it is possible to develop similarities and differences that can reinforce the study of each separately. Of course in a monograph of this size only a selection of topics can be covered, and it is hoped that the present volume will act as a supplement to the standard works in these fields. An extensive bibliography is given at the end of the book.

Some of the digital methods presented in this book have found wide-spread applications. The multichannel prediction-error filter, and more particularly modified versions of it which are still designed on the same general principle of computing the filter coefficients on the basis of a noise sample preceding the event, are in use in digital seismic detection systems for the surveillance of underground nuclear explosions. The single and multichannel recursive methods for the solution of normal equations involving correlation matrices are widely employed. The method of predictive decomposition (deconvolution) is being used extensively as a practical digital data processing method to eliminate unwanted reverberations on seismic records taken in the exploration for oil and gas. As a result large water-covered areas of the globe can now be explored for oil and gas, which hitherto had to be classified as 'no-good' seismic regions because of the destructive interference of water reverberations with the desired reflection signals from deep geological structures.

Digital computer programs for many of the methods given in this book may be found in S.M. Simpson (1966), *Time-Series Computations in FORTRAN and FAP*, Volume 1, *A Program Library*, Addison-Wesley Publishing Co., Reading, Massachusetts; and E.A. Robinson (1966), *Collection of FORTRAN Programs for Filtering and Spectral Analysis of Single Channel Time Series*, Supplement No. 1 of *Geophysical Prospecting*, Volume 14, European Association of Exploration Geophysicists, The Hague, The Netherlands.

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ENDERS A. ROBINSON

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FOREWORD

by Markus Båth

Seismology – like geophysics in general – works on a broad front, extending over observations in nature (seismic records), laboratory investigations, and theoretical studies. It is an applied mathematical-physical science, in which the major steps forward have been closely connected to impacts from outside, that is from other sciences where methods and techniques already have reached a high level of perfection. In the history of seismology there are many scientists who were specialists in an adjacent field as well, and thus were able to work in a border-line field – that is where new ideas are frequently born and new results come out. The present book by Professor Enders Robinson is another instance of such cross-fertilization of two fields – statistical communications theory and seismology – and is the most extensive and thorough treatment which has appeared so far in this border region.

The similarities as well as the dissimilarities between seismic signals and radar signals are clearly emphasized, and what is particularly gratifying to seismologists is the clear exposition of statistical communications theory and filtering techniques, and their applications to seismological records. With partial exception for seismic exploration, much more sophisticated methods have been developed earlier in radar observations than in seismology. It is only in the last few years – after the installation of seismic array stations – that more developed techniques are coming into more general use.

A record obtained from an earthquake or an explosion bears the signature of the source properties, the path properties, and the seismograph response. In addition, seismic signals are always immersed in noise to varying degrees. In any seismological study it is of the greatest significance to distinguish clearly between these various factors and between signal and noise, a task which can sometimes be very difficult with traditional methods, but is much facilitated by methods described in this book.

Filtering and correlation techniques, which are extensively described here, offer flexible and efficient methods to improve seismological studies, as for example to detect and extract weak signals in the presence of noise or for signal enhancement and prediction. Filtering exists and has always done so in seismic recordings but in a less flexible way. Path filtering is naturally an unavoidable factor and beyond human influence. Instumental filtering (due to seismograph response) is another kind of filtering, which has been used efficiently

to separate various frequencies from each other using a series of seismographs: short-period, medium-period, long-period. But even this represents a certain amount of compromise between what is practicable and what is required for a complete coverage of the wave spectrum. Also, once set up, it represents an inflexible system, which is only seldom changed. Filtering on the final records has so far been used only to a very limited extent, but obviously opens up new possibilities for seismological research – not only by means of array-station records but also on the more traditional station records. In addition, such methods are of the greatest significance in any nuclear surveillance system.

The human element still plays a very significant role in the reading and interpretation of seismic records, but the adaptive seismic array systems, described in the last chapter, certainly point to the future. Starting from the seismic wave source, everything is automatic up to the point when a seismic record is obtained (except when the source is a man-made explosion). This has naturally been so since the birth of instrumental seismology. The next step, the reading and interpretation of the record and the reporting in bulletins, is where the human element comes in and still is the most competent worker. In earthquake seismology, usually only visual inspection of records – with no special filtering – is used, at least in routine readings. Still more so, readings of records are too often left to unqualified people, in spite of the fact that enormous funds have been invested in the instrumental installations. Once the reports have reached one or several of the seismological world data centres, the next step takes over, that is the computation of source parameters. Since around 1960 this has been done on electronic computers, and thus represents an essentially automatic step. It is of interest to state that the intermediate step is very much dependent on the human element, and even if progress for its elimination is hopeful, it will certainly take some time before a trained and qualified seismologist can be dispensed with in the reading of records. But the new methods would be highly welcome, as they imply increased precision, increased reliability, and increased speed of data sampling. Digital seismographs may be considered as a step in this direction, but have so far only been operated on an experimental basis at a few institutes.

From this point of view and from the outlook expressed in the last chapter of this book, it is tempting to speculate on the future of seismological recordings. For rapid information – within the lapse of minutes – both of earthquakes and of explosions, it would be suitable to have a world-wide system of adaptive seismic array stations, which would automatically transmit data to certain centres where all pertinent source parameters were continuously calculated and then distributed.

FOREWORD

by Anthony F. Gangi

This book demonstrates the similarities that exist between radar and seismology. They both use waves to detect anomalies in the medium through which the waves propagate. The major problem in these two disciplines is the extraction of the desired return signal from the ever-present noise. This is the problem to which this book is directed. To extract the desired signal from the noise, or to discriminate against the noise, full advantage must be taken of the differences in the temporal and spatial characteristics between the signal and the noise. The Fourier transform is a powerful mathematical tool in delineating these differences. The concepts of matched filtering, shape filtering, auto-correlation, cross-correlation, convolution, and deconvolution are important processes in the discrimination problem, and these processes are clarified in terms of the Fourier transform.

Data sampling is used in both radar and seismology. The use of sampled data systems leads to the z-transform which is derived from the Fourier transform. The similarities between time-sampled data and seismic or antenna arrays become readily apparent when it is realized that the signals from the arrays correspond to space-sampled data. The mathematics, fundamental constraints, and fundamental principles are then easily seen to be the same for these two seemingly different situations. This naturally leads to the concept of spatial filtering which is analogous to temporal filtering.

The digitization of the sampled data leads to the concept of digital signals and digital filtering. This digitization is performed so that the data may be processed on a digital computer. The flexibility, speed of operation, and data-handling capability of digital computers make them invaluable in the signal-noise discrimination problem. The mathematics of convolution and the Fourier transform, originally applied to continuous signals or functions, are readily modified for application to digital signals and digital filtering techniques.

The flexibility of digital computers makes it possible to consider adaptive filtering techniques in the discrimination problem. The adaptive techniques automatically take advantage of the differences in the properties of the signals and the noise. Thus the filtering process or discrimination process is made a function of both time and space. The importance of adaptive processing has been realized for some time, but only recently have techniques developed to the point where it is possible to use adaptive processing in practice. Adaptive processing is described in this book and its underlying principles are enunciated.

Those interested in the techniques of signal and noise discrimination and digital data processing will find Professor Robinson's book of great value in their studies.

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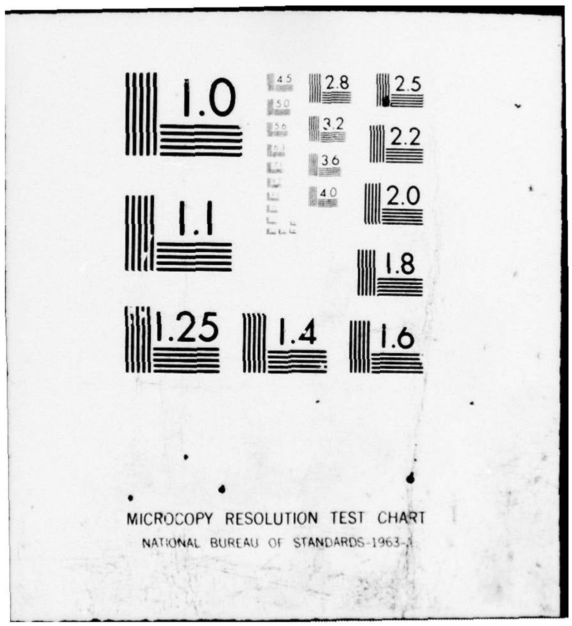
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1967

Yu. A. Rozanov

STATIONARY RANDOM PROCESSES

Translated by

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HOLDEN-DAY

San Francisco. Cambridge. London. Amsterdam

TRANSLATOR'S PREFACE

This translation has benefited, in comparison with the Russian original, by the inclusion of certain improved results connected with the factorization of rational spectral densities, which were communicated to me by the author, as well as by the correction (by author and translator) of various minor errors, mainly of a typographical nature.

It is a pleasure to thank the author for his considerable assistance in these matters.

A. Feinstein

AUTHOR'S PREFACE

In recent years new developments have taken place in the theory of stationary random processes. Many papers, devoted to linear forecasting of multi-dimensional stationary (in the wide sense) processes, have brought this portion of the theory close to a final definitive form. Considerable attention has been focused on various kinds of ergodic properties of stationary (in the strict sense) processes which arise in connection with the applicability of the central limit theorem to these processes. This book, which originated in a course of lectures on the theory of stationary processes given by me at Moscow University in 1959-1960, is devoted to these questions.

This book is intended for the mathematically qualified reader, but the basic results which it contains (particularly those relating to rational spectral densities) should also be intelligible to the engineering reader who is interested in applications of the theory of stationary processes.

In writing this book I have benefited considerably from various remarks by my friends. To all of them my sincere thanks.

I consider it my pleasant duty to express here my deep gratitude to my teacher Andrei Nikolaevich Kolmogorov.

Yu. A. Rozanov

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SPECTRAL ANALYSIS and its applications

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Preface

Time series analysis is now widely used in many branches of engineering, the physical sciences and economics. One important aspect of time series analysis is spectral analysis, which is concerned with the splitting up of time series into different frequency components. Applications of spectral analysis cover a wide range of problems, for example, the effect of wave oscillations on the vibration of ships and the influence of disturbances or noise on the performance of electrical guidance systems and chemical reactors.

This book has been designed primarily for post-graduate engineers, since most of the applications of spectral analysis have, in fact, been made by engineers and physicists. One of the difficulties faced by users or potential users of spectral analysis is that most of the theory has been developed by statisticians during the last fifteen years. Unfortunately, much of this literature is difficult to read. Hence it is felt that a book directed mainly toward engineers is long overdue. However, we hope this book will appeal to a much wider audience, including mathematicians, statisticians, economists, physicists and biologists.

One of the difficulties we have encountered in writing this book is that, whereas spectral analysis involves the use of sophisticated statistical techniques, many engineers lack knowledge of elementary statistics. This is true even of electrical engineers, some of whom possess considerable knowledge of probability theory. For example, the Wiener theory of prediction and control shows that an optimum filter or control system can be designed if various spectra associated with the signal and noise in the system are known. However, little attention is paid in books on control theory to the very important practical question of how to *estimate* these spectra from *finite lengths of record*. It is with such problems that we shall be concerned in this book.

To provide a gradual introduction to time series estimation problems, we have been forced in the earlier chapters to deal with elementary statistical problems. This may distract mathematical and statistical readers, but in view of our experience in expounding these ideas to engineers, we feel that

a self-contained introduction, which includes most of the statistical ideas needed later on in the book, is necessary. Those readers who are familiar with the material of Chapters 2, 3 and 4 can, of course, start at Chapter 5.

Chapter 1 is devoted to a brief outline of the territory covered and to a description of the kind of problems which can be solved using spectral analysis. Chapter 2 deals with the important ideas of Fourier analysis and is basic to what follows. Most of this is well known to engineers but is brought together here in a form oriented toward spectral analysis. In Chapter 3 we introduce some basic notions in probability theory which are fundamental to subsequent chapters. Chapter 4 consists of an introduction to many important ideas in statistical inference and includes a discussion of the sampling distribution approach to estimation theory, the theory of least squares and a brief reference to likelihood inference. Not all of this material is necessary for an understanding of the spectral techniques discussed later in the book, and engineering readers may wish to omit the latter part of this chapter at first reading. The most relevant parts of this chapter, as far as spectral analysis is concerned, are the sections on the sampling distribution approach to estimation theory and the theory of least squares. The latter is one of the most important weapons in the statistician's armory, and in our experience it is widely misunderstood among engineers.

Chapter 5 contains some of the simpler ideas in the theory of stochastic processes, for example, stationarity, the autocorrelation function and moving average-autoregressive processes. Methods for estimating autocorrelation functions and parameters in linear processes are described and illustrated by examples. In Chapter 6, the ideas of Fourier analysis and stochastic processes are brought together to provide a description of a stationary stochastic process by means of its spectrum. It is shown how Fourier methods need to be tailored to estimate the spectrum from finite lengths of record. The sampling properties of spectral estimators are then derived, and the important notion of smoothing of these estimators is introduced. Chapter 7 contains many simulated and practical examples of spectral estimation and gives a systematic method, called window closing, for deciding the amount of smoothing required.

In Chapter 8 the ideas of Chapters 5–7 are extended to pairs of time series, leading to the definition of the cross correlation function, the cross spectrum and the squared coherency spectrum. Chapter 9 is devoted to estimating the cross spectrum and the notion of aligning two series. Cross spectral analysis is applied in Chapter 10 to estimating the frequency response function of a linear system. Finally, we consider in Chapter 11 the spectral analysis of a vector of several time series and the estimation of the frequency response matrix of a linear system.

This book has been written at a time when there is much active work in this area and when much experience has still to be gained in the application

of spectral methods. Nevertheless, it is felt that enough has been achieved already to warrant an attempt. It is hoped that the book will provide applied scientists and engineers with a comprehensive and useful handbook for the application of spectral analysis to practical time series problems, as well as proving useful as a post-graduate textbook.

We are greatly indebted to Professor K. N. Stanton of the School of Engineering, Purdue University, for making available the power-station data used in later chapters and to Professor H. J. Wertz of the University of Wisconsin for helpful suggestions regarding computer programs. We are very grateful to Mr. A. J. A. MacCormick of the Statistics Department of the University of Wisconsin and also Mr. M. J. McClellan of the U.S. Army Mathematics Research Center, University of Wisconsin, for writing and running some of the computer programs. We also thank Mr. MacCormick, and Mr. A. S. Alavi of the University of Lancaster, for checking through the manuscript.

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Multichannel Time Series Analysis with Digital Computer Programs

Revised Edition 1978

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Preface

This book is one member of the Holden-Day Series in Time Series Analysis. The entire series will give a comprehensive presentation of time series analysis from several different points of view. As a result, we have concentrated on certain aspects of time series analysis; these aspects are unified under the general philosophy presented in the *Nonmathematical Introduction*. Briefly, our approach is to link time series methods closely with the empirical and theoretical evidence provided by the subject matter under investigation. Because the writer's personal experience has been largely in geophysics, many of the methods are presented in their relationship to the analysis of geophysical time series recorded in the seismic exploration for oil. However, these methods can be quite readily adapted to analyze other physical time series.

Chapter 1 introduces some of the basic ideas and computer programs which are used in subsequent chapters. Chapter 2 presents single-channel time series analysis with computer programs. Chapter 3 is a digression into the physical mechanism of wave propagation in layered media in order to bring out the physical significance of such concepts as minimum delay, autocorrelation, and prediction-error operators. Chapter 4 treats the subject of polynomials with matrix coefficients; this material forms the mathematical basis for much of the theory of multichannel time series analysis. Chapter 5 presents methods of digital filtering and spectral analysis of multichannel time series with computer programs. Chapter 6 gives an extensive treatment of multichannel Wiener filtering and signal enhancement with computer programs.

All of the computer programs in this book are written in Fortran IV. Hence, a reader can key-punch them and use them directly for time series analysis on any digital computer which accepts Fortran. In fact, many significant results will be made by people with practical needs who make use of these computer programs to analyze actual time series data, provided that they also make full use of their own individual know-how and ingenuity. The

present book does presuppose that the potential user of these programs has knowledge of the Fortran language equivalent to that given in any of the elementary texts in Fortran programming or the first term of a Fortran programming course at the university. Many practical people may want to read the last chapter first, for it contains some of the more interesting programs.

In this book we have used the symbol i for $\sqrt{-1}$, instead of j, which is commonly used by electrical engineers. However, we have followed the engineer's usage of the superscript asterisk * to denote complex conjugate, mainly because it is easier to type and print than the overscore. We have used the superscript ^{T} to denote matrix transpose. We have not used the superscript dagger † for complex-conjugate transpose, but instead we have used the more explicit ^{*T}. In the text, we often write an isolated Fortran statement, such as:

```
CALL NLOGN (N, X, SIGN).
```

The punctuation (usually a period or comma) following such a statement, as the period following the above statement, is not part of the statement, but is part of the English sentence in which the statement appears.

Professor Norbert Wiener (1894-1964) was generous with his ideas and always accessible to the younger staff at M.I.T. His original help in 1954 and 1955 in going over his work on multiple time series with me had a decided influence on this book.

For their help and encouragement I want to express my sincere thanks to Professor Markus Båth, Professor Jon Claerbout, Professor Emanuel Parzen, Dr. Sven Treitel, Dr. Ralph Wiggins, and Professor Herman Wold.

Tulsa, Oklahoma
January 1978

ENDERS A. ROBINSON

Nonmathematical Introduction

Scientific models

In time series analysis the dualism between observation and theory is recognized as fundamental. This dualism between empirical evidence and theoretical analysis is embodied in the notion of the scientific model. To develop a model it is necessary to simplify. For example, one may model an industrial firm as a system where the investments by the firm in new equipment and inventory represent the inputs to the system, and the returns on the investments represent the outputs of the system. This idealized economic situation represents a scientific model which makes description and explanation of the actual situation easier. In practice, however, such a simple model has limited usefulness, for it is usually the case that there are many competing firms, and as a result the interactions between firms must be taken into account. Hence, an ensemble of different models is required to handle the complex situations encountered in economics. Thus, the approach of scientific model-building must be pluralistic; there can never be a unique, all-embracing scientific model. The variety of types of scientific models is almost unlimited for the situations encountered in time series analysis.

A scientific model of necessity must always be a compromise between simplicity and reality. It is interesting to look at some of the factors which have influenced the evolutionary trends of the models used in time series analysis. The most important factor, of course, has been the development of statistical communication theory, which dates from the work of Norbert Wiener at M.I.T. during World War II. At one time, statistical communication theory was regarded as a potential unifier which would provide solutions of problems within many scientific disciplines in terms common to each of them. However, this ideal unification has not materialized since unifying principles have the tendency to become highly specialized subjects in themselves, thereby compounding rather than easing the problems of bridging many scientific problem areas with common methods. However, this tendency

is not the only difficulty associated with unification. The most important one is that despite the similarities which exist among many scientific fields, fundamental differences also exist. Hence a detailed understanding of the physical, economic, or biological processes within each field always remains an important requirement. As a result, the models must take these factors into account, and hence we see that we must introduce meaningful subject-matter information into models in order to account for the problems we face in any scientific field.

New ideas take the form of new scientific models. There are verbal models, mathematical models, and physical models. A verbal model is simply a word description of the scientific situation. A mathematical model is one that reduces essential features of the situation into the form of mathematical equations. A physical model is a physical apparatus which represents an analog scaled-down version of the actual situation. Physical models have a long history in the sciences. In the past, mathematical models of necessity were greatly simplified. However, with the advent of the digital computer, the range of possibilities of mathematical models is greatly enhanced. No longer will mathematical models be essentially theoretical. With the large memory capacity of the new generation of computers, it is now possible to build mathematical models which indeed make use of the large body of empirical evidence which exists as data in any science. In the future, we can expect mathematical models to represent in a more equitable way both theory and observation, that is, both theoretical analysis and empirical evidence. In this way, the gap between theory and observation can be bridged, thereby forming a framework to support new ideas and new methods.

In making scientific interpretations, we are confronted in many cases not with direct problems, but with indirect or inverse problems. For example, in he physical sciences we are faced with the problems of inferring the location ind properties of some unknown physical mass such as a submarine or buried re body from physical measurements which can only be taken at a distance. n the biological sciences, we wish to determine things such as the state of ealth of the internal organisms of an astronaut from measurements which an be conveniently monitored while he is engaged in his normal duties. In conomics, we wish to make inferences about the economic forces from measred prices and quantities. It is due to this inverse nature of many scientific oblems that scientific interpretation is difficult. Knowledge and experience addition to numerical data are needed to achieve meaningful solutions. In der to attack the interpretation problem a flexible mathematical model is quired whose outcomes can be readily tested against the measurements ovided by direct observation. If the agreement is not good, then the model ust be altered until the values do correspond at least to within satisfactory nits. The model becomes useful when it is able to reproduce the more signifit results of the actual situation. However, it may be possible to find other

models which work just as accurately. At this point, any attempts to resolve the ambiguity must lie in bringing in new information. In particular, meaningful and relevant practical information should be incorporated because the final model will be interpreted in practical terms.

In summary, a scientific model represents the embodiment of theory and observation. Of necessity, it must be a compromise between simplicity and reality. There will never be one all-encompassing model; there must be many kinds of models to fit different situations. Above all, a model must incorporate practical information because the final interpretation will be based on practical needs.

Digital data processing

Why have we discussed scientific models? The reason is that a model is required before there can be any data processing. The best data processing is that which is done with the best model. Data processing of scientific evidence to achieve practical ends, therefore, needs good scientific modeling.

In this book, we are concerned with time series data, that is, data which is ordered in time. Moreover, we are concerned with digital processing of the time series data, that is, processing which is done by a digital computer.

Let us now take an example from geophysics to make these concepts more meaningful. The word *convolution* means "folding." When a signal is passed into a filter, the output is given as the convolution of the signal with the impulse response function of the filter. Likewise, when a signal is passed into the earth, the output seismogram represents the convolution of the input signal with the response function of the earth. The word *deconvolution* means "unfolding." A deconvolution operator is one which produces the inverse operation to the given convolution operator. The objective in deconvolving seismic evidence with a computer, then, is to unfold it from its original heterogeneous complexities into its simpler components so that the interpreter can see the basic structuring of the data. Thus, a deconvolution operator is one which separates specific results produced on a seismogram by earth transmission effects and data collection systems from pertinent information as to deep-reflected energy which is needed in making the geologic interpretations. For example, system deconvolution would separate out effects of the data collection systems, reverberation deconvolution would separate out the effects of reverberations, and multiple reflection deconvolution would separate out the multiple reflections from the primary reflections. In the broad sense, deconvolution may be defined as any operation that unfolds the recorded data in such a way that it separates redundant or unwanted information from desired or wanted information.

In this context, we see that the problem of deconvolving data differs from the classical problem of separating signal from noise. For example, in the

classic radar situation, the received signal is masked by noise. This noise is white noise generated within the receiver and, as such, is completely independent of the signal. Hence, in such a case filters are designed to destroy as much of the noise as possible, for the noise contains no useful information about the radar environment. In contrast, nearly all the information recorded on a seismic trace represents meaningful information about the total seismic environment, and the problem is to unfold the recorded seismogram into simpler components so that an interpreter can see which energy is due to primary events and which is due to ghost events, multiple events, or to other causes. The digital processing method of deconvolution represents a way of unfolding this information.

In a similar argument, one can say that nearly all the data recorded in economic time series of share prices, commodity futures, quantities, and other economic indicators represent *meaningful* information. The problem then is to unfold this data into simpler components which can be interpreted. Nearly all the measurements taken on a human body represent valid information; the problem is to deconvolve the time series data into less complex components for medical interpretation.

Deconvolution

The concept of deconvolution is a key idea which underlies many of the developments in this book. For this reason, we want to discuss further the over-all concept in this nonmathematical introduction. For any process of deconvolution, we must construct a model which takes into account both known factors and unknown factors. The known factors are used in deterministic aspects of the model, and the unknown factors are incorporated in statistical aspects of the model. Let us now describe a basic model. The desired information consists of primary events or innovations. These innovations are unpredictable: they occur at random times and with random amplitudes. Attached to each innovation is a response. The response represents the reverberations or repercussions generated by the innovation, and each response persists over a relatively long time span before it damps out. The recorded time series consists of the sum of all the primary events with their attached responses. Because the primary events are not well separated in time, the high degree of overlap among the various responses completely masks out the onset of each primary event. As a result, a study of the recorded time series does not reveal the separate events, but only shows a highly mixed conglomeration of all the events which cannot be interpreted.

We have now described a model which generates time series data. These data represent useful information, but, as such, they are not in a form which can be interpreted. In order to interpret the data we must unfold or deconvolve the data. The basic problem then is to unfold the time series so as to separate the primary events from the reverberations. Separately these components can be interpreted, for they represent the underlying structure which makes up the recorded time series.

For example, with the process of deconvolution one can answer questions such as these: Is a certain movement of economic prices the fortuitous result of several responses adding together in phase, or does it represent some new primary event? From all the recorded data on the human body, how can we sort out any essential changes in the human state of health from effects due to previous conditions? In radar and sonar, how can environmental reverberation and clutter be separated from the true signal?

Redundancy

How does one devise a deconvolution process for the given model? In principle, every deconvolution process is based upon the cornerstone of information theory, namely the concept of redundancy.

To bring these ideas out more clearly, let us make a small digression into the realm of information theory. The measure of information in information theory is called *entropy*. The ratio of the entropy per symbol to the maximum value it could have while still restricted to the same symbols is called the *relative entropy* ratio. One minus the relative entropy ratio is called the *redundancy*. Thus, the redundancy represents the factor by which the average lengths of messages are increased beyond the minimum necessary to transmit the desired information. *The fundamental theorem of information theory* states: A source of information can be encoded in such a way that, when transmitted over a noisy channel, the rate of transmission may approach the channel capacity with the probability of error as small as desired. This theorem is based upon the use of redundancy as a "noise-reducing" mechanism.

Let us explain. One way of reducing errors due to noise is to repeat the signal. For example, telegraph companies usually repeat numbers and names, and the sender himself sometimes repeats critical words. More generally, in many scientific situations, the symbols or parts that make up the message are linked among themselves. This linking makes the symbols partly redundant and thereby provides a check on the accuracy of the message as it is received. For this reason, every code has some overlap or redundancy. That is, its symbols do not all add their full weight of innovation, but instead they add a certain amount of internal confirmation. In this way, the reliability or accuracy of transmission over a noisy channel can be increased by means of redundancy. The source is encoded in a redundant manner, which in effect means that parts of a message tell us things already partly known. The receiver can then make use of this redundancy in order to predict the nearby

succeeding parts of a message to a considerable extent. As a result, the actual reception of a message gives partly a verification or correction of the preceding prediction, in addition to any completely new ideas.

As we have just seen, redundancy is the partial or complete repetition of message content. Redundancy forms the basis upon which encoding methods can be devised to increase the transmission rate up to the channel capacity. That is, we can approach the absolute capacity of a channel, to any tolerance we fix, by putting our messages in a code whose redundancies are appropriate to the form of the noise. The redundancies show up upon reception and allow us to correct all of the errors except for the given proportion. Hence, redundancy gives the code a structure or skeleton which resists the distortion of its individual symbols. Many of the engineering aspects of information theory are concerned with building systems in which adequate redundancy is incorporated into the system in order to overcome noise.

On the other hand, in many observational systems, nature itself has built enough redundancy into the system in order to overcome noise. However, this redundancy causes so much overlapping and mutual interference that is is not possible to interpret the raw records provided by nature. The process of deconvolution in effect separates the new information from the redundant information as time progresses. This separation makes interpretation possible, for both kinds of information (new and redundant) have value in understanding the basic mechanisms involved. The new information consists of the primary events or innovations, and the redundant information the attached responses or repercussions.

Minimum delay model as a basis for deconvolution

As we have seen, a model should represent both theory and observation. It is the judicious combination of both theory and observation which makes deconvolution possible.

First, let us consider theory. By theoretical reasoning, we must arrive at some partial knowledge of the innovations and the reverberations. For example, the model for seismic deconvolution is what is called a *minimum-delay model*. The model is based upon two theoretical hypotheses, namely:

(1) The deterministic hypothesis that the reverberation attached to each primary event has the same minimum-delay shape.
(2) The statistical hypothesis that the primary events are randomly spaced in time and have random amplitudes.

The deterministic hypothesis is based upon the fact that the reverberations are caused by the vibrating physical system made up of fixed geological layers near the surface. In Chapter 3, we show that wave propagation through layered system produces a minimum-delay response. Since each primary event passes through the same layered system at the surface, each has the same minimum-delay response attached to it. The statistical hypothesis is based on the fact that the primary events are caused by reflections from deep geological beds within the earth. These deep layers were laid down in geologic time in an unsystematic way, and thus the primary events produced by them are randomly spaced in time and amplitude.

Next, observation must be combined with theory. The empirical evidence is in the form of the time series, namely, the seismic trace. The autocorrelation of the time series is computed. The autocorrelation is a method of averaging which averages out, or destroys, the unsystematic elements of the time series but preserves features of the systematic elements. The features preserved are those contained in the autocorrelation (or, equivalently, the amplitude spectrum) of the reverberating system. We must now combine this empirical amplitude spectrum with the theoretical minimum-phase spectrum. This essential step is possible because of the minimum-delay nature of the model. Hence, using both theory and observation together, we are able to reconstruct the minimum-delay reverberating system. Having the reverberation waveform, we can then design the deconvolution operator which removes the reverberations from the seismic trace; the output is the deconvolved seismic trace consisting of the primary events only.

Such models upon which deconvolution processes are based can be only approximately true; however, the existing models are sufficiently accurate to make the deconvolution processes based upon them worthwhile in an economic sense. For a time, people will be satisfied with these results, but as time goes on, we will want to develop better processes. These better processes must be based on broader models. Hopefully, our expanded knowledge will allow us to push forward the frontier between the deterministic and the statistical; that is, some factors which we had to treat statistically before can in the future be reduced to deterministic analysis, and factors which were completely neglected before can in the future be at least encompassed within some statistical framework. As we do this, our whole frame of reference will become larger.

For example, in seismic interpretation, a geophysicist in the 1920's looked at a single trace at a time, then at a record of up to twenty-four traces in the 1930's and 1940's, then at cross sections of single coverage in the 1950's, and cross sections of multiple coverage in the 1960's. Now in the mapping of salt domes and stratigraphic traps, one must look at three-dimensional models. With this perspective, present-day reverberation-elimination deconvolution acts on a single trace; multiple-elimination deconvolution acts on several traces. We envisage that the next generation of deconvolution procedures will act on entire cross sections and will encompass three spatial dimensions. Factors which appeared random on the basis of a small amount of data will find deterministic explanations, and factors which were missed en-

-irely on the basis of limited data will be amenable to statistical treatment on the basis of the increased data samples. The frontier will be pushed forward. The new data processing techniques must explain local space-time effects in terms of a regional model: this regional model in turn must make full use of all the pertinent information which can be derived from the data.

Time structure of data

Let us now look at the time aspects of data processing. A time series is a function of time, which can be either historical time or real time. Associated with the concept of time is the concept of prediction, for prediction represents the concept of searching for a time structuring or ordering of events which occur as time progresses. Originally it was felt that although the problem of prediction is important in many branches of science that involve real time records, such as meteorology, it is not a significant problem in the analysis of historical time records. That is, it was held that the concept of prediction does not really enter into the problem of the analysis of a time series recorded sometime in the past. As a result, it was almost universally believed that any investigation of the predictive properties of such time series would not be fruitful. However, this seeming disadvantage has been turned into an advantage by refocusing attention from the predictions to the prediction errors. If we look at the space-time complex, we see that the new information entering a time series as time progresses is from the primary (innovation) events, and hence unpredictable from the previous events. Reverberations, repercussions, and multiple reflections, however, are predictable events resulting from primary events already accounted for. Hence, a prediction-error operator represents a means of separating the unpredictable new events from the predictable reverberations, repercussions, and multiples. In Section 2.10, we discuss the prediction-error operator in the form of the deconvolution operator commonly used for the elimination of reverberations on seismic traces; however, much more general prediction-error operators can be devised to sort out new information from prior information on time series in many branches of science. This unfolding of information into the form of the dynamic structure as well as the series of innovations ordered in time and space is the essence of the deconvolution process. Deconvolution is based on the concept that new information, or innovations, are not predictable from the past and, hence, are exhibited as prediction errors. The best deconvolution of data is done from the best model, that is, the model which represents in the best way the scientific situation under consideration. The best final interpretation results from the best recognition of the validity of the model that is revealed after the unfolding process.

We want to use the term deconvolution in its larger sense, namely, as the process of unfolding the information on a time series into the predictable

components such as reverberations, repercussions, and multiples and into the unpredictable components which represent the successive innovations. The next significant item in the deconvolution process which we want to discuss may be called the time-space scale of the deconvolution. On the one hand, we have deconvolution in the small where we consider at one time only a single time series. On the other hand, we have deconvolution in the large where we consider at one time ensembles of time series produced on a regional scale.

Deconvolution in the large

Let us give a specific example taken from seismology of deconvolution in the large. Let our model be the normal-incidence synthetic seismogram which includes all multiples, as described in Chapter 3.

This synthetic seismogram is based upon a parallel layered model of the earth in which plane waves travel along ray paths normal to the interfaces between the layers. The solution of the wave equation at each interface leads to the definition of a reflection coefficient associated with that interface. The reflection coefficient has several properties well known to every geophysicist. The most characteristic property is that the reflection coefficient physically cannot exceed unity in magnitude. If the reflection coefficient is equal to unity in magnitude, then the interface is a perfect reflector, and no transmission takes place through the interface. The best example of such a situation is provided by the interface representing the surface of the water in marine exploration: the reflection coefficient of the water surface is -1, or nearly so. In land exploration, the ground-to-air interface reflects nearly 100% of the energy coming up to it. Other interfaces can have high reflection coefficients; for example, it is not unusual for the base of the surface weathered layer to have a reflection coefficient of 0.5. Some interfaces, of course, have low reflection coefficients. Hence, our model is made up of a sequence of parallel interfaces where each has a reflection coefficient associated with it. Hence, in the discrete layer case, the reflection amplitudes are determined by the reflection coefficients.

Let us now consider the case of a layered system for which the upper interface is a perfect reflector. Such a case is realized in marine seismic exploration, for then the first layer is the water layer, and the surface of the water acts as a perfect reflector. Let us assume that the source is a single positive spike. This source results in a seismic trace recorded at the surface which represents all direct reflections and multiple reflections resulting from the layered system at depth.

As shown in Chapter 3, this synthetic trace itself may be regarded as one-half of an autocorrelation function. Also, from Section 2.8, we know that a prediction-error operator, or so-called deconvolution operator, of any

length may be computed from any given autocorrelation function. Hence, we can compute deconvolution operators of various lengths directly from the recorded seismic trace.

Let us now take a seismic trace and, treating it as an autocorrelation function, compute deconvolution operators from it of successively increasing length. For example, if our discrete time spacing were 4 msec, we would compute deconvolution operators of length 4 msec, 8, 12, 16, 20, 24, 28, etc., up to, say, 4000 msec long. As we show in Chapter 3, we may regard this ensemble of deconvolution operators of successively increasing lengths as a means of portraying the structure of the layered earth. For example, if we convolve the deconvolution operator of length, say, 600 msec, with the trace, we obtain as output the wave motion at depth 300 msec. If we convolve the deconvolution operator of length 1400 msec with the trace, we obtain as output the wave motion at depth 700 msec. Hence, by this deconvolution process, we can obtain a three-dimensional picture of the wave motion in the earth from the two-dimensional observations at the surface. The potential value of such processes of deconvolution in the large lies in the possibility of obtaining a very detailed picture of the subsurface structure (in three dimensions) from observations taken at the surface (in two dimensions).

Thus, the computer has allowed the scientist to refocus his attention on the entire process of prediction, namely, to find time structures along time series by unfolding them into their components and seeing how the time events interlock and fit together.

Computers: digital and environmental

Today the working scientist in the normal course of commercial work is exposed to large amounts of data, and, in fact, his interpretations of these data represent research of great theoretical as well as practical value. However, in order to do meaningful research, one must have time—time to think and time to try new ideas. Usually time is strictly limited though, and hence the scientist must extract as much information as he can from the data in the limited time available to him. However, such working under a constant deadline often defeats the ultimate objective, for vital information might be missed or misinterpreted simply because of the economic pressure of getting the job done quickly. As a result, if scientists are to fullfill their expanding role, they must, of necessity, expand in number as well as talent.

Scientists are continually expanding in talent, for today a scientist can do things that were impossible just a few years ago. In the next few years, the scientist will expand his capabilities many-fold over what it is today by the expanded use of machines: data-generating machines, recording machines, computing machines, and data-display machines. This is indeed a stimulating prospect and one to encourage new people into science, thereby expanding

the number of scientists as well. Until recently, most scientific data were recorded by fixed equipment, so the scientist was completely dependent in his results on the analog equipment supplied to him. Except by an involved trial-and-error process in which new equipment was designed over the years, the scientist had very limited control over the type of operation he could apply to the data in any given instance. With the advent of digital recording equipment, the scientist can now record a much more detailed band of information; he can then digitally operate on this information with a general purpose digital computer. As a result, he can design operations on the basis of the data at hand and on his present needs and ideas; he can alter or change his methods in a matter of minutes by a programming change; he can incorporate fresh ideas into the analysis and see the results immediately.

At present, this ideal situation has not fully materialized, for usually there is still a significant time gap (the so-called *turn-around time*) between the original data collection and the final interpretation in most scientific work. However, with the new generation of computers and with such programming advances as time-sharing and the new computer languages, it is within our grasp to overcome the problem of turn-around time in most cases.

The computing machine will have a tremendous impact on science. Computer science represents an entirely new scientific discipline. Computer science is the science of the future. The computer by its handling of great masses of information allows man to spend more of his time on the edge of the unknown and thus lets him apply his unique talents to the field of the unknown. This ability of the computer to save time and manipulate large amounts of information has given new scope to the application of man's talents.

In this sense, the computer must be incorporated in the total environment of science. As an example, again taken from seismology, the earth itself can be used as a computer. Through the use of high-powered electro-mechanical vibrators, signals of specific types can be impressed in the earth. By the use of directional detectors, signals can be picked up, and coded signals can then be fed back into the earth under the control of a digital computer on the surface, all in real time. Hence, the earth itself becomes a computing machine which can effect the deconvolution process as the time series are recorded. Thus, we can use the earth itself to analyze the data from the earth. In this way, we can utilize seismic recording systems more efficiently; our equipment will not be idle a good part of the time as it is now, but will be used at great efficiency by incorporating the environment itself into our computing system. In the biomedical sciences, the human body itself will be used as a computer under control of digital computers, and thereby the final data recorded will already be deconvolved and hence in a form ready for interpretation. Thus, man can make his enviroment a computer which in turn is linked to the digital computer. Energy is recycled into the environmental computer in accordance with the needs determined by sampling the original

energy return from the environment. It is possible for man to see the results of his own ideas at the time of recording by such use of the environment as a computer. The information from the environment can be unfolded or deconvolved during the recording process itself, for man can link the digital computer to the environmental computer to acquire such information.

In closing this introduction, let us be careful to point out that such use of computers does not mean that the machine decides for us all the vital decisions to be made, for at the present time it would be very difficult or impossible to provide the machine with all the information which is germane to such decisions. The chief use of the machine is to prepare the information and to display it to the human interpreter who then incorporates into the decision such highly significant information as only the human mind has available. The logical development of such a use of the computer is as follows. As more and more of the unknown factors become better understood, they can be incorporated into the machine programs, and hence the scientific interpreter will always be able to concentrate his attention on the unknown factors. The computer will be used as a machine to always keep the scientist at the forefront of human knowledge, for it is at the edge of the unknown that the human mind has the comparative advantage over the computer and will produce the significant contributions.

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1969

SPECTRAL METHODS IN ECONOMETRICS

George S. Fishman

Harvard University Press
Cambridge, Massachusetts 1969

Preface

Time series analysis is a branch of statistics whose scope and method have broadened considerably in the past quarter century. Among the developments of this period, the spectral approach, which concerns the decomposition of a time series into frequency components, has become a principal tool of analysis in such diverse fields as communications engineering, geophysics, oceanography, and electroencephalography. Spectral methods have also been applied to the analysis of economic time series, and it is evident that the recent extension of these methods to the estimation and testing of distributed lag models will attract more interest in applying this approach to econometrics.

This book describes spectral methods and their use in econometrics. It is intended as an introduction for graduate students and econometricians who wish to familiarize themselves with the topic.

The literature often emphasizes the ability of the spectral approach to reveal periodic or almost periodic components in a time series. While this information is of interest to the econometrician, it does not justify his learning more than a meager amount about the spectral approach to time series analysis, but a number of other reasons do justify learning about it in detail. The spectral approach has conceptual advantages for conveying information about interdependent events and also has desirable properties for working with sample data. In addition, it often suggests models for explaining the time-varying behavior of economic phenomena, it often enables one to determine the effects of transformations on variables, it often simplifies hypothesis testing, and

it often permits more efficient use of sample data in the estimation of time-domain model parameters than do more traditional methods.

Mathematically, time series may be regarded as a topic in Hilbert space analysis. If we were to use this approach, the conciseness of the notation would unfortunately obscure many intuitive analogies that prove so helpful in acquainting readers with a new topic. Alternatively, the benefits of spectral methods would not emerge clearly in a predominantly heuristic description. This is especially true in studying simultaneous distributed lag equations. To convey the significance of spectral methods for econometric analyses properly, some compromise is necessary.

A study of this book therefore requires a knowledge of probability theory and difference equations for univariate and bivariate time series analysis, and also a knowledge of matrix analysis and multivariate statistical analysis for multivariate time series analysis. A familiarity with operations on complex variables is also helpful, but not essential.

In writing this book, I have attempted to include as many of the recent advances in spectral methods as have appeared in the open literature and that seem relevant for econometrics. The rate at which new work appears has made this attempt difficult, as have the many new questions that arise whenever spectral methods are extended. For example, the sample properties of the fast Fourier transform technique remain to be worked out in detail. The theory of testing hypotheses related to distributed lag models is also incomplete. Wherever these new advances are mentioned, I have emphasized the transitional state of their development to encourage the reader to watch for future developments.

Thanks are due Marc Nerlove, who read the entire manuscript and suggested numerous improvements, to Bennett Fox, who checked a number of the mathematical results, and to Murray A. Geisler and Albert Madansky, who encouraged me to write the book.

I would also like to thank Jacque McMullen, Nancy Gibson, and Marie Spaugy for their excellent typing of several versions of the manuscript, Doris Dong for preparing the artwork, and Roberta Schneider and Willard Harriss for their editorial assistance.

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George S. Fishman

Santa Monica, California
January 1969

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1970

INTRODUCTION TO STOCHASTIC CONTROL THEORY

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1970

ACADEMIC PRESS New York and London

PREFACE

The object of this book is to present stochastic control theory — analysis, parametric optimization and optimal stochastic control. The treatment is limited to linear systems with quadratic criteria. It covers discrete time as well as continuous time systems.

The first three chapters provide motivation and background material on stochastic processes. Chapter 4 is devoted to analysis of dynamical systems whose inputs are stochastic processes. A simple version of the problem of optimal control of stochastic systems is discussed in Chapter 6; this chapter also contains an example of an industrial application of this theory. Filtering and prediction theory are covered in Chapter 7, and the general stochastic control problem for linear systems with quadratic criteria is treated in Chapter 8.

In each chapter we shall first discuss the discrete time version of a problem. We shall then turn to the continuous time version of the same problem. The continuous time problems are more difficult both analytically and conceptually. Chapter 6 is an exception because it deals only with discrete time systems.

There are several different uses for this volume:

- A short applications-oriented course covering Chapter 1, a survey of Chapter 2, Sections 1, 2 and 3 of Chapters 3 and 4, Sections 1 and 4 of Chapter 5, Chapter 6, and a survey of Chapters 7 and 8.
- An introductory course in discrete time stochastic control covering the sections mentioned above, and also Section 2 of Chapter 5, Sections 1-5 of Chapter 7, and Sections 1-6 of Chapter 8.
- A course in stochastic control including both discrete time processes as well as continuous time processes covering the whole volume.

The prerequisites for using this book are a course in analysis, one in probability theory (preferably but not necessarily covering the elements of stochastic processes), and a course in dynamical systems which includes frequency response as well as the state space approach for continuous time and discrete time systems. A reader who is acquainted with the deterministic theory of optimal control for linear systems with quadratic criteria

will get a much richer understanding of the problems discussed, although this knowledge is not absolutely required in order to read this book.

This work is an expansion of the notes from lectures delivered to various industrial and academic audiences between 1962–1969. A preliminary version was given in seminars in 1963 at the IBM Research Laboratories in San Jose, California and Yorktown Heights, New York. An expanded version was presented during 1964 and 1965 at the IBM Nordic Laboratory, the Royal Institute of Technology, and the Research Institute of National Defense, all located in Stockholm, Sweden. Part of this material has been used in graduate courses at the Lund Institute of Technology, Sweden, since 1965. The complete manuscript was presented as a graduate course in stochastic control at the Lund Institute of Technology during the 1968–1969 academic year.

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TIME SERIES ANALYSIS forecasting and control

Revised Edition 1976

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HOLDEN-DAY

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Preface

Much of statistical methodology is concerned with models in which the observations are assumed to vary independently. In many applications dependence between the observations is regarded as a nuisance, and in planned experiments, *randomization* of the experimental design is introduced to validate analysis conducted as if the observations were independent. However, a great deal of data in business, economics, engineering and the natural sciences occur in the form of *time series* where observations are *dependent* and where the nature of this dependence is of interest in itself. The body of techniques available for the analysis of such series of dependent observations is called *time series analysis*.

Spectral analysis, in the frequency-domain, comprises one class of techniques for time series analysis, but we shall say very little here about that important subject. This book is concerned with the building of stochastic (statistical) models for discrete time series in the time-domain and the use of such models in important areas of application. Our objective will be to derive models possessing maximum simplicity and the minimum number of parameters consonant with representational adequacy. The obtaining of such models is important because:

(1) They may tell us something about the nature of the system generating the time series;

(2) They can be used for obtaining *optimal forecasts* of future values of the series;

(3) When two or more related time series are under study, the models can be extended to represent dynamic relationships between the series and hence *to estimate transfer functions*;

(4) They can be used to derive *optimal control policies* showing how a variable under one's control should be manipulated so as to minimize disturbances in some dependent variable.

The ability to forecast optimally, to understand dynamic relationships between variables and to control optimally is of great practical importance. For example, optimal sales forecasts are needed for business planning, transfer function models are needed for improving the design and control of process plant and optimal control policies are needed to regulate important process variables, both manually and by the use of on-line computers. Over the last ten years the authors have worked with real data arising in economics and industry and, by trial and error, and by a long sequence of interactions between theory and practice, have attempted to select, adapt, and develop practical techniques to fulfill such needs. This book is the fruit of these labors.

The approach adopted is, first, to discuss a class of models which are sufficiently flexible to describe practical situations. In particular, time series are often best represented by *nonstationary* models in which trends and other psuedo-systematic characteristics which can change with time are treated as statistical rather than as deterministic phenomena. Furthermore, economic and business time series often possess marked *seasonal* or periodic components themselves capable of change and needing (possibly nonstationary) seasonal statistical models for their description.

The process of model *building*, which is next discussed, is concerned with relating such a class of statistical models to the data at hand and involves much more than model fitting. Thus, *identification* techniques, designed to suggest what particular kind of model might be worth considering, are developed first and make use of the autocorrelation and partial autocorrelation functions. The *fitting* of the identified model to a time series using the likelihood function can then supply maximum likelihood estimates of the parameters or, if one prefers, Bayesian posterior distributions. The initially fitted model will not, necessarily, provide adequate representation. Hence *diagnostic checks* are developed to detect model inadequacy, to suggest appropriate modifications and thus, where necessary, to initiate a further iterative cycle of identification, fitting and diagnostic checking.

When forecasts are the objective, the fitted statistical model is used directly to generate optimal forecasts by simple recursive calculation. In particular, this model completely determines whether the forecast projections should follow a straight line, an exponential curve, and so on. In addition, the fitted model allows one to see exactly how the forecasts utilize past data, to determine the variance of the forecast errors, and to calculate limits within which a future value of the series will lie with a given probability.

When the models are extended to represent dynamic relationships, a corresponding iterative cycle of identification, fitting and diagnostic checking is developed to arrive at the appropriate transfer function-stochastic model. In the final section of the book, the stochastic and transfer function models developed earlier are employed in the construction of feedforward and feedback control schemes.

The applications given in this book are by no means exhaustive and it is hoped that the examples presented will enable the reader to adapt the techniques to his own problem. In particular the difference equations used to represent transfer functions and stochastic phenomena may be employed as building blocks which when appropriately fitted together can simulate a wide variety of the systems occurring in engineering, business and economics. Furthermore the principles of model building which are discussed and illustrated have very general application.

AN OUTLINE OF THE BOOK

This book is set out in the following parts (from time to time, a vertical line has been inserted in the left margin to indicate material which may be omitted in the first reading):

Introduction and Summary (Chapter 1)

This chapter is an informal and highly condensed outline of topics discussed, defined and more fully explained in the main body of the text. It is intended as a broad mapping of areas to be subsequently explored, and the student may wish to refer back to it as later chapters are read.

Part I Stochastic models and their forecasting (Chapters 2, 3, 4 and 5)

After some basic tools of time series analysis have been discussed in Chapter 2, an important class of linear stochastic models is introduced in Chapters 3 and 4 and their properties discussed. The immediate introduction of forecasting in Chapter 5 takes advantage of the fact that the form of the optimal forecasts follows at once from the structure of the stochastic models discussed in Chapter 4.

Part II Stochastic model building (Chapters 6, 7, 8 and 9)

Part II of the book describes an iterative model-building methodology whereby the stochastic models introduced in Part I, are related to actual time series data. Chapters 6, 7 and 8 describe, in turn, the processes of model identification, model estimation, and model diagnostic checking. Chapter 9 illustrates the whole model building process by showing how all these ideas may be brought together to build seasonal models and how these models may be used to forecast seasonal time series.

Part III Transfer function model building (Chapters 10 and 11)

In Chapter 10 transfer function models are introduced for relating a system output to one or more system inputs. Chapter 11 discusses methods for transfer function-noise model identification, estimation and diagnostic checking. The chapter ends with a description of how such models may be used in forecasting.

Part IV Design of discrete control schemes (Chapters 12 and 13)

In these two chapters we show how the stochastic models and transfer function models previously introduced may be brought together in the design of simple feedforward and feedback control schemes.

A first draft of the book was produced in 1965 and subsequently was issued in 1966 and 1967 as Technical Reports Nos. 72, 77, 79, 94, 95, 99, 103, 104, 116, 121 and 122 of the Department of Statistics, University of Wisconsin, and Nos. 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 13 of the Department of Systems Engineering, University of Lancaster. The work has involved a great deal of research, which has been partially supported by the Air Force Office of Scientific Research, United States Air Force, under AFOSR Grants AF-AFOSR-1158-66 and AF-49 (638) 1608 and also by the British Science Research Council. We are grateful to Professor E. S. Pearson and the Biometrika Trustees for permission to reprint condensed and adapted forms of Tables 1, 8 and 12 of *Biometrika Tables for Statisticians*, Vol. 1, edited by E. S. Pearson and H. O. Hartley, and to Dr. Casimer Straikowski for permission to reproduce and adapt three figures from his doctoral thesis, University of Wisconsin, 1968. The authors are indebted to George Tiao, David Mayne, Emanuel Parzen, David Pierce, Granville Wilson and Donald Watts for suggestions for improving the manuscript, to John Hampton, Granville Wilson, Elaine Hodkinson and Patricia Blant for writing the computer programs described at the end of the book and also to them and to Dean Wichern, David Bacon and Paul Newbold for assistance with the calculations. Finally, we are glad to record our thanks to Hiro Kanemasu, Paul Newbold, Larry Haugh, John MacGregor and Granville Wilson for careful reading and checking of the manuscript, to Carole Leigh and Mary Esser for their care and patience in typing the manuscript and to Meg Jenkins for the initial draft of the diagrams.

G. E. P. Box, Madison, U.S.A.
G. M. Jenkins, Lancaster, U.K.
June 1969

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1970

Time Series
Data Analysis and Theory

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PREFACE

The initial basis of this work was a series of lectures that I presented to the members of Department 1215 of Bell Telephone Laboratories, Murray Hill, New Jersey, during the summer of 1967. Ram Gnanadesikan of that Department encouraged me to write the lectures up in a formal manner. Many of the worked examples that are included were prepared that summer at the Laboratories using their GE 645 computer and associated graphical devices.

The lectures were given again, in a more elementary and heuristic manner, to graduate students in Statistics at the University of California, Berkeley, during the Winter and Spring Quarters of 1968 and later to graduate students in Statistics and Econometrics at the London School of Economics during the Lent Term, 1969. The final manuscript was completed in mid 1972. It is hoped that the references provided are near complete for the years before then.

I feel that the book will prove useful as a text for graduate level courses in time series analysis and also as a reference book for research workers interested in the frequency analysis of time series. Throughout, I have tried to set down precise definitions and assumptions whenever possible. This undertaking has the advantage of providing a firm foundation from which to reach for real-world applications. The results presented are generally far from the best possible; however, they have the advantage of flowing from a single important mixing condition that is set down early and gives continuity to the book.

Because exact results are simply not available, many of the theorems of the work are asymptotic in nature. The applied worker need not be put off by this. These theorems have been set down in the spirit that the indicated

asymptotic moments and distributions may provide reasonable approximations to the desired finite sample results. Unfortunately not too much work has gone into checking the accuracy of the asymptotic results, but some references are given.

The reader will note that the various statistics presented are immediate functions of the discrete Fourier transforms of the observed values of the time series. Perhaps this is what characterizes the work of this book. The discrete Fourier transform is given such prominence because it has important empirical and mathematical properties. Also, following the work of Cooley and Tukey (1965), it may be computed rapidly. The definitions, procedures, techniques, and statistics discussed are, in many cases, simple extensions of existing multiple regression and multivariate analysis techniques. This pleasant state of affairs is indicative of the widely pervasive nature of the important statistical and data analytic procedures.

The work is split into two volumes. This volume is, in general, devoted to aspects of the linear analysis of stationary vector-valued time series. Volume Two, still in preparation, is concerned with nonlinear analysis and the extension of the results of this volume to stationary vector-valued continuous series, spatial series, and vector-valued point processes.

Dr. Colin Mallows of Bell Telephone Laboratories provided the author with detailed comments on a draft of this volume. Professor Ingram Olkin of Stanford University also commented on the earlier chapters of that draft. Mr. Jostein Lillestöl read through the galleys. Their suggestions were most helpful.

I learned time series analysis from John W. Tukey. I thank him now for all the help and encouragement he has provided.

D.R.B.

Berkeley, California
June, 1974

CONTENTS

1

THE NATURE OF TIME SERIES AND THEIR FREQUENCY ANALYSIS

1.1 INTRODUCTION

In this work we will be concerned with the examination of r vector-valued functions

$$\begin{bmatrix} X_1(t) \\ \cdot \\ \cdot \\ \cdot \\ X_r(t) \end{bmatrix} \tag{1.1.1}$$

where $X_j(t)$, $j = 1, \ldots, r$ is real-valued and t takes on the values $0, \pm 1, \pm 2, \ldots$. Such an entity of measurements will be referred to as an r **vector-valued time series.** The index t will often refer to the time of recording of the measurements.

An example of a vector-valued time series is the collection of mean monthly temperatures recorded at scattered locations. Figure 1.1.1 gives such a series for the locations listed in Table 1.1.1. Figure 1.1.2 indicates the positions of these locations. Such data may be found in World Weather Records (1965). This series was provided by J. M. Craddock, Meteorological Office, Bracknell. Another example of a vector-valued time series is the set of signals recorded by an array of seismometers in the aftermath of an earthquake or nuclear explosion. These signals are discussed in Keen et al (1965) and Carpenter (1965). Figure 1.1.3 presents an example of such a record.

$$\mathbf{A}'(\lambda) = \mathbf{A}(\lambda)\mathbf{f}_{XX}(\lambda)^{1/2}$$
$$\mathbf{B}'(\lambda) = \mathbf{B}(\lambda)\mathbf{f}_{YY}(\lambda)^{1/2}.$$

By Schwarz's inequality the coherency is

$$\leqslant \frac{\mathbf{B}'(\lambda)\mathbf{f}_{YY}(\lambda)^{-1/2}\mathbf{f}_{YX}(\lambda)\mathbf{f}_{XX}(\lambda)^{-1}\mathbf{f}_{XY}(\lambda)\mathbf{f}_{YY}(\lambda)^{-1/2}\overline{\mathbf{B}'(\lambda)}^{\tau}}{\mathbf{B}'(\lambda)\overline{\mathbf{B}'(\lambda)}^{\tau}}$$
$$\leqslant \mu_j[\mathbf{f}_{YY}(\lambda)^{-1/2}\mathbf{f}_{YX}(\lambda)\mathbf{f}_{XX}(\lambda)^{-1}\mathbf{f}_{XY}(\lambda)\mathbf{f}_{YY}(\lambda)^{-1/2}]$$

for $\mathbf{B}'(\lambda)$ orthogonal to $\mathbf{V}_1(\lambda), \ldots, \mathbf{V}_{j-1}(\lambda)$ the first $j - 1$ latent vectors of $\mathbf{f}_{YY}^{-1/2}\mathbf{f}_{YX}\mathbf{f}_{XX}^{-1}\mathbf{f}_{XY}\mathbf{f}_{YY}^{-1/2}$ by Exercise 3.10.26. Expression (10.3.25) indicates that $\mathbf{B}_j(\lambda)$ is as indicated in the theorem; that $\mathbf{A}_j(\lambda)$ achieves equality follows by inspection.

Proof of Theorem 10.3.3 Because the latent roots of $\mathbf{f}_{YX}\mathbf{f}_{XX}^{-1}\mathbf{f}_{XY}$ are simple for all λ, its latent roots and vectors are real holomorphic functions of the entries, see Exercises 3.10.19–21. Expressions (10.3.28) and (10.3.29) now follow from Theorem 3.8.3. Expression (10.3.30) follows from (10.3.26) to (10.3.29).

Proof of Theorem 10.3.4 Because the latent roots of $\mathbf{f}_{YY}^{-1/2}\mathbf{f}_{YX}\mathbf{f}_{XX}^{-1}\mathbf{f}_{XY}\mathbf{f}_{YY}^{-1/2}$ are simple for all λ, its latent roots and vectors are real holomorphic functions of its entries; see Exercises 3.10.19 to 3.10.21. Expressions (10.3.33) and (10.3.34) now follow from Theorem 3.8.3. That the spectral density is (10.3.36) either follows from Theorem 10.3.1 or by direct computation.

Proof of Theorem 10.4.1 This follows as did the proof of Theorem 9.4.1 with the exception that the perturbation expansions of the proof of Theorem 10.2.6 are now used.

Proof of Theorem 10.4.2 This follows from the above perturbation expansions in the manner of the proof of Theorem 10.2.6.

Proof of Theorem 10.4.3 The $\hat{\mu}_j$, $\hat{\mathbf{A}}_j$, and $\hat{\mathbf{B}}_j$ are continuous functions of the entries of (10.4.25). The theorem consequently follows from Theorem 7.3.3 and Theorem P5.1.

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1970

Multiple Time Series

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Preface

The subject of time series analysis has intimate connections with a wide range of topics, among which may be named statistical communication theory, the theory of prediction and control, and the statistical analysis of time series data. The last of these is to some extent subsidiary to the other two, since its purpose, in part at least, must be to provide the information essential to the application of those theories. However, it also has an existence of its own because of its need in fields (e.g., economics) in which at present well-developed, exact theories of control are not possible. It is with the third of these topics that this book is concerned. It extends beyond that in two ways. The first and most important is by the inclusion in the first half of the book of a fairly complete treatment of the underlying probability theory for second-order stationary process. Although this theory is for the most part classical and available elsewhere in book form, its understanding is an essential preliminary to the study of time series analysis and its inclusion is inevitable. I have, however, included a certain amount of material over and above the minimum necessary, in relation, for example, to nonlinear filters and random processes in space as well as time. The statistical development of this last subject is now fragmentary but may soon become important.

The second additional topic is the theory of prediction, interpolation, signal extraction, and smoothing of time series. The inclusion of this material seems justified for two reasons. The first arises from the understanding that the classical "Wiener-Kolmogoroff" theories give of the structure of time series. This understanding is needed, in part at least, for statistical developments (e.g., identification problems and problems associated with the relation between the eigenvalues of the covariance matrix and the spectrum). The second reason is that these developments are becoming important to people who are statisticians concerned with time series (e.g., in missile

trajectory estimation and economics). Of course, some of the more practically valuable work here is recent and would require a separate volume for an adequate treatment, but some indications concerning it are needed.

There is one other characteristic that a modern book on time series must have and that is the development of the theory and methods for the case in which multiple measurements are made at each point, for this is usually the case.

Having decided on the scope of the book, one must consider the manner in which the material will be presented and the level of the presentation. This book sets out to give the theory of the methods that appear to be important in time series analysis in a manner that it is hoped will lead finally to an understanding of the methods as they are to be used. On the whole it presents final formulas but often does not discuss computational details and it does not give computer programs. (For the most part the methods discussed are already programmed and these programs are available.) With minor exceptions numerical examples are not given. It is not a book on "practical time series analysis" but on the theory of that subject. There is a need for books of the first kind, of course, but also of this second kind, as any time series analyst knows from requests for references to a definitive discussion of the theory of this or that topic. The level of presentation causes problems, for the theory is both deep and mathematically unfamiliar to statisticians. It would probably be possible to cover the underlying probability theory more simply than has been done by making more special assumptions (or by making the treatment less precise). To make the book more accessible a different device has been used and that is by placing the more difficult or technical proofs in chapter appendices and starring a few sections that can be omitted. It is assumed that the reader knows probability and statistics up to a level that can be described as familiarity with the classic treatise *Mathematical Methods of Statistics* by Harald Cramér. A mathematical appendix which surveys some needed elementary functional analysis and Fourier methods has been added.

Some topics have not been fully discussed, partly because of the range of my interests and partly because of the need to keep the length of the book within reasonable bounds. I have said only a small amount about the spectra of higher moments. This is mainly because the usefulness of this spectral theory has not yet been demonstrated. (See the discussion in Chapter II, Section 8.) Little also has been said about nonstationary processes, and particularly about their statistical treatment. This part of the subject is fragmented at the moment. Perhaps, of necessity, it always will be. A third omission is of anything other than a small discussion of "digitized" data (e.g., "clipped signals" in which all that is recorded is whether the phenomenon surpassed a certain intensity). There is virtually no discussion

of the sample path behavior of Gaussian processes, for this subject has recently been expertly surveyed by Cramér and Leadbetter (1967) and its inclusion here is not called for. I have also not discussed those inference procedures for point processes based on the times of occurrence of the events in the process (as distinct from the intervals between these times). This has recently been surveyed by Cox and Lewis (1966). Finally, the second half of the book (on inference problems) discusses only the discrete time case. This is justified by the dominance of digital computer techniques.

I have not attempted to give anything approaching a complete bibliography of writing on time series. For the period to 1959 a very complete listing is available in Wold (1965). It is hoped that the references provided herein will allow the main lines of development of the subject to the present time to be followed by the reader.

I have many people to thank for help. The book developed from a course given at The Johns Hopkins University, Baltimore, Maryland, and an appreciable part of the work on it was supported by funds from the United States Air Force. The book's existence is due in part to encouragement from Dr. G. S. Watson. Dr. C. Rhode at Johns Hopkins and R. D. Terrell, P. Thomson, and D. Nicholls at the Australian National University have all read parts of the work and have corrected a number of errors in its preliminary stages. The typing was entirely done, many times over, by Mrs. J. Radley, to whom I am greatly indebted.

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Canberra, Australia
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Contents

† These topics are special and may be omitted.

$\lambda_i \mu_j$. If λ_i is of multiplicity m_i and μ_j of multiplicity n_j then $\lambda_i \mu_j$ has multiplicity $m_i n_j$. This may be seen by taking A, B as realized by matrices and reducing A, B to Jordan canonical form by the transformation $A \to PAP^{-1}$, $B \to QBQ^{-1}$. In this canonical form PAP^{-1} and QBQ^{-1} have null elements below the main diagonal and their eigenvalues in appropriate multiplicity, in the main diagonal. Thus if $A \otimes B$ is taken in the first matrix realization then $A \otimes B \to (PAP^{-1}) \otimes (QBQ^{-1}) = (P \otimes Q)(A \otimes B)(P \otimes Q)^{-1}$ again reduces $A \otimes B$ to upper triangular form and evidently $A \otimes B$ has $\lambda_i \mu_j$ with multiplicity $m_i n_j$.

It follows, in particular, that $\det (A \otimes B) = \{\det (A)\}^q \{\det (B)\}^p$ and $\operatorname{tr} (A \otimes B) = \operatorname{tr} (A) \operatorname{tr} (B)$.

These definitions may be extended to an arbitrary number of tensor factors so that we may form $\mathfrak{X}_1 \otimes \mathfrak{X}_2 \otimes \cdots \otimes \mathfrak{X}_r$. These may be defined inductively. Similarly we define $A_1 \otimes A_2 \otimes \cdots \otimes A_r$ if A_j acts in $\mathfrak{X}_j$ as a linear transformation. We shall not list the extensions to $r > 2$ of the properties listed above since these are all fairly obvious. For example, if the eigenvalues of A_j are $\lambda_{j,i}$, with multiplicity $n_{j,i}$ then the eigenvalues of $A_1 \otimes A_2 \otimes \cdots \otimes A_r$ and their multiplicities are

$$\prod_{j=1}^{r} \lambda_{j,i(j)}, \qquad \prod_{j=1}^{r} n_{j,i(j)},$$

where all possible choices $i(1), \ldots, i(r)$ are allowed.

It may be observed from the abstract definition first given that $\mathfrak{X} \otimes \mathfrak{Y}$ and $\mathfrak{Y} \otimes \mathfrak{X}$ are isomorphic as vector spaces and similarly there is an isomorphism between $\mathfrak{X}_1 \otimes \mathfrak{X}_2 \otimes \cdots \otimes \mathfrak{X}_r$ and the tensor product of the $\mathfrak{X}_j$ taken in any other order. If $\mathfrak{X} \otimes \mathfrak{Y}$ (and $\mathfrak{Y} \otimes \mathfrak{X}$) are realized as spaces of $p \times q$ (and $q \times p$) matrices then the isomorphism of $\mathfrak{X} \otimes \mathfrak{Y}$ with $\mathfrak{Y} \otimes \mathfrak{X}$ is clearly attained through $C \in \mathfrak{X} \otimes \mathfrak{Y} \leftrightarrow C' \in \mathfrak{Y} \otimes \mathfrak{X}$. This isomorphism defines a corresponding isomorphism of the algebra of linear operators in $\mathfrak{X} \otimes \mathfrak{Y}$ into the algebra of linear operators in $\mathfrak{Y} \otimes \mathfrak{X}$ and clearly under this correspondence $A \otimes B \leftrightarrow B \otimes A$. The generalization to more than two factors is evident.

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1971

The Statistical Analysis of Time Series

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Preface

In writing a book on the statistical analysis of time series an author has a choice of points of view. My selection is the mathematical theory of statistical inference concerning probabilistic models that are assumed to generate observed time series. The probability model may involve a deterministic trend and a random part constituting a stationary stochastic process; the statistical problems treated have to do with aspects of such trends and processes. Where possible, the problem is posed as one of finding an optimum procedure and such procedures are derived. The statistical properties of the various methods are studied; in many cases they can be developed only in terms of large samples, that is, on information from series observed a long time. In general these properties are derived on a rigorous mathematical basis.

While the theory is developed under appropriate mathematical assumptions, the methods may be used where these assumptions are not strictly satisfied. It can be expected that in many cases the properties of the procedures will hold approximately. In any event the precisely stated results of the theorems give some guidelines for the use of the procedures. Some examples of the application of the methods are given, and the uses, computational approaches, and interpretations are discussed, but there is no attempt to put the methods in the form of programs for computers.

This book grew out of a graduate course that I gave for many years at Columbia University, usually for one semester and occasionally for two semesters. By now the material included in the book cannot be covered completely in a two-semester course; an instructor using this book as a text will select the material that he feels most interesting and important. Many exercises are given. Some of these are applications of the methods described; some of the problems are to work out special cases of the general theory; some of the exercises fill in details in complicated proofs; and some extend the theory.

Besides serving as a text book I hope this book will furnish a means by which statisticians and other persons can learn about time series analysis without resort to a formal course. Reading this book and doing selected exercises will lead to a considerable knowledge of statistical methodology useful for the analysis of time series. This book may also serve for reference. Much material which has not been assembled together before is presented here in a fairly coherent fashion. Some new theorems and methods are presented. In other cases the assumptions of previously stated propositions have been weakened and conclusions strengthened.

Since the area of time series analysis is so wide, an author must select the topics he will include. I have described in the introduction (Chapter 1) the material included as well as the limitations, and the Table of Contents also gives an indication. It is hoped that the more basic and important topics are treated here, though to some extent the coverage is a matter of taste. New methods are constantly being introduced and points of view are changing; the results here can hardly be definitive. In fact, some material included may at the present time be rather of historical interest.

In view of the length of this book a few word of advice to readers and instructors may be useful in selecting material to study and teach. Chapter 2 is a self-contained summary of the methods of least squares. It may be largely redundant for many statisticians. Chapters 3 and 4 deal with models with independent random terms (known sometimes as "errors in variables"); some ideas and analysis are introduced which are used later, but the reader interested mainly in the later chapters can pass over a good deal (including much of Sections 3.4, 4.3, and 4.4). Autoregressive processes, which are useful in applications and which illustrate stationary stochastic processes, are treated in Chapter 5; Sections 5.5 and 5.6 on large-sample theory contain relevant theorems, but the proofs involve considerable detail and can be omitted. Statistical inference for these models is basic to analysis of stationary processes "in the time domain." Chapter 6 is an extensive study of serial correlation and tests of independence; Sections 6.3 and 6.4 are primarily of theoretical statistical interest; Section 6.5 develops the algebra of quadratic forms and ratios of them; distributions, moments, and approximate distributions are obtained in Sections 6.7 and 6.8, and tables of significance points are given for tests. The first five sections of Chapter 7 constitute an introduction to stationary stochastic processes and their spectral distribution functions and densities. Chapter 8 develops the theory of statistics pertaining to stationary stochastic processes. Estimation of the spectral density is treated in Chapter 9; it forms the basis of analyzing stationary processes "in the frequency domain." Section 10.2 extends regression analysis (Chapter 2) to stationary random terms; Section

10.3 extends Chapters 8 and 9 to this case; and Section 10.4 extends Chapter 6 to the case of residuals from fitted trends. Parts of the book that constitute units which may be read somewhat independently from other parts are (i) Chapter 2, (ii) Chapters 3 and 4, (iii) Chapter 5, (iv) Chapter 6, (v) Chapter 7, and (vi) Chapters 8 and 9.

The statistical analysis of time series in practical applications will also invoke less formal techniques (which are now sometimes called "data analysis"). A graphical presentation of an observed time series contributes to understanding the phenomenon. Transformations of the measurement and relations to other data may be useful. The rather precisely stated procedures studied in this book will not usually be used in isolation and may be adapted for various situations. However, in order to investigate statistical methods rigorously within a mathematical framework some aspects of the analysis are formalized. For instance, the determination of whether an effect is large enough to be important is sometimes formalized as testing the hypothesis that a parameter is 0.

The level of this book is roughly that of my earlier book, *An Introduction to Multivariate Statistical Analysis*. Some knowledge of matrix algebra is needed. (The necessary material is given in the appendix of my earlier book; additional results are developed in the text and exercises of this present book.) A general knowledge of statistical methodology is useful; in particular, the reader is expected to know the standard material of univariate analysis such as t-tests and F-statistics, the multivariate normal distribution, and the elementary ideas of estimation and testing hypotheses. Some more sophisticated theory of testing hypotheses, estimation, and decision theory that is referred to is developed in the exercises. [The reader is referred to Lehmann (1959) for a detailed and rigorous treatment of testing hypotheses.] A moderate knowledge of advanced calculus is assumed. Although real-valued time series are treated, it is sometimes convenient to write expressions in terms of complex variables; actually the theory of complex variables is not used beyond the simple fact that $e^{i\theta} = \cos\theta + i\sin\theta$ (except for one problem). Probability theory is used to the extent of characteristic functions and some basic limit theorems. The theory of stochastic processes is developed to the extent that it is needed.

As noted above, there are many problems posed at the end of each chapter except the first which is the introduction. Solutions to these problems have been prepared by Paul Shaman. Solutions which are referred to in the text or which demonstrate some particularly important point are printed in Appendix B of this book. Solutions to most other problems (except solutions that are straightforward and easy) are contained in a Solutions Manual which is issued as a separate booklet. This booklet is available free of charge by writing to the publisher.

I owe a great debt of gratitude to Paul Shaman for many contributions to this book in matters of exposition, selection of material, suggestions of references and problems, improvements of proofs and exposition, and corrections of errors of every magnitude. He has read my manuscript in many versions and drafts. The conventional statement that an acknowledged reader of a manuscript is not responsible for any errors in the publication I feel is usually superfluous because it is obvious that anyone kind enough to look at a manuscript assumes no such responsibility. Here such a disclaimer may be called for simply because Paul Shaman corrected so many errors that it is hard to believe any remain. However, I admit that in this material it is easy to generate errors and the reader should throw the blame on the author for any he finds (as well as inform him of them).

My appreciation also goes to David Hinkley, Takamitsu Sawa, and George Styan, who read all or substantial parts of the manuscript and proofs and assisted with the preparation of the bibliography and index. There are many other colleagues and students to thank for assistance of various kinds. They include Selwyn Gallot, Joseph Gastwirth, Vernon Johns, Ted Matthes, Emily Stong Myers, Emanuel Parzen, Lloyd Rosenberg, Ester Samuel, and Morris Walker as well as Anupam Basu, Nancy David, Ronald Glaser, Elizabeth Hinkley, Raul Mentz, Fred Nold, Arthur V. Peterson, Jr., Cheryl Schiffman, Kenneth Thompson, Roger Ward, Larry Weldon, and Owen Whitby. No doubt I have forgotten others. I also wish to thank J. M. Craddock, C. W. J. Granger, M. G. Kendall, A. Stuart, and Herman Wold for use of some material.

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1971

RANDOM DATA: ANALYSIS AND MEASUREMENT PROCEDURES

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PREFACE

This book is an extensive revision and replacement for the authors' early book, *Measurement and Analysis of Random Data*, 1966. Approximately 50 percent of the original material has been rewritten or deleted and replaced by new material. These changes reflect the technical advances that have taken place in the last five years as well as an increased awareness of pertinent matters gained through the further experience of the authors. Specifically, a broader discussion appears on statistical errors in random data analysis. An entirely new chapter has been introduced to integrate the general requirements for data acquisition, recording, preparation, qualification, and processing. The discussions of digital data analysis procedures have been greatly expanded to cover the more recent analysis techniques made feasible by the availability of fast Fourier transform algorithms. Discussions of transient and multidimensional random processes are now included. Finally, a number of illustrative examples involving actual physical data have been added to support theoretical developments. The illustrations are largely restricted to aerospace and automotive applications since these are the fields of most recent concern to the authors. The general techniques, however, are applicable to data common in many other fields including meteorology, oceanography, seismology, communications, nuclear processes, and biomedical research.

The emphasis in this new book is on the practical aspects of random data analysis and measurement procedures, with special attention to the interrelationships of the various technical disciplines involved. As before, the book is written with the primary intent of providing a convenient reference for practicing engineers and scientists. The secondary intent of providing a specialized textbook for students has been augmented by the addition of problem sets at the end of each chapter. The reader is assumed to have a basic knowledge of probability theory, statistics, and transform methods of applied mathematics.

Summaries of chapter contents appear at the beginning of each chapter. In brief, Chapters 1 through 4 present a review of basic theoretical background material needed for the developments in later chapters. Basic descriptive properties of random data are outlined in Chapter 1 while physical system response properties are reviewed in Chapter 2. Pertinent mathematical

and statistical theory is summarized in Chapters 3 and 4. This review material is followed in Chapters 5 and 6 by extensive developments and formulations of input-output relationships and statistical errors in measured data. Chapter 7 outlines the overall procedures for random data acquisition and processing. Detailed procedures for analog and digital data analysis are presented in Chapters 8 and 9. The final Chapter 10 discusses some advanced ideas and procedures relevant to nonstationary, transient, and multidimensional data.

We wish to acknowledge the many contributions to this book by former associates in Measurement Analysis Corporation and Digitek Corporation. We also thank those government agencies, industrial companies, and individuals who supported our work. A special appreciation is given to Engineering Extension, University of California, Los Angeles, and to other organizations, who sponsored our presentation of short courses on this subject matter. Our final thanks extends to Teresa Piersol and Lucinda Bendat for their help in preparing the manuscript.

Los Angeles, California
July 1971

JULIUS S. BENDAT
ALLAN G. PIERSOL

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1973

Time-Series

Sir Maurice Kendall

M.A., Sc.D., F.B.A.

Second edition 1976

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Preface to the First Edition

In the last thirty years the theory of time-series has been transformed into a new subject. In part this is due to the introduction of probabilistic ideas into what was formerly treated deterministically; in part it is attributable to the power of the electronic computer, which has removed the obstacles imposed by the extensive and tedious calculations involved in most time-series studies. There has, nevertheless, tended to appear a rift between sophisticated theory and practical application, and although there exists an extensive literature in scientific and professional journals there are few books which attempt to treat the subject in its entirety for the benefit of the practising statistician.

That is my reason for writing this book. It aims to present the basic ideas and techniques of the subject, with as much exemplification as space will permit and a determination not to let the mathematics multiply beyond necessity. I have tried to make it the sort of book that I would like to have had put in my hand when I first became interested in time-series many years ago.

I am indebted to Professor Dudley J. Cowden and the Director of the School of Business Administration, University of North Carolina, for permission to reproduce Appendix Table A; to Professor James Durbin and the Editors of *Biometrika* for permission to reproduce Appendix Tables B; to Professor C.W.J. Granger for permission to reproduce Fig. 8.4; and to Dr D.J. Reid for permission to reproduce Fig. 9.1.

M.G.K.

London
February, 1973

Preface to the Second Edition

A number of misprints and obscurities have been removed and some references added in this edition, which otherwise follows the same lines as the first.

M.G.K.

London
December, 1975

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APPENDIX A

Weights for fitting polynomial trends

The following tables are extracted by permission from Dudley J. Cowden, *Weights for fitting polynomial secular trends*, Technical Paper No. 4, School of Business Administration, University of North Carolina, 1962. Professor Cowden gives values up to N (the extent of the moving average) = 25, and also values for even N.

Except in one or two early tables, the tables give the weights required to fit at one end of the series, those for the other end being given by symmetry. For example, for fitting a straight line (a simple moving average) to nine points the weights for the first point are $\frac{1}{45}$ [17, 14, 11, 8, 5, 2, −1, −4, −7]. Those for the second are $\frac{1}{180}$ [56, 47, 38, 29, 20, 11, 2, −7, −16] and so on. Conversely, for the last four points, the weights are (reading the table upwards) for the last but three, [8, 11, 14, 17, 20, 23, 26, 29, 32] and so on.

The columns headed 0 give the weights required to extrapolate the fitting one unit beyond the end of the observed series.

The sums in the last row but one are the sums of the integral weights given in the table.

The final row in the table is the square root of the error-reducing factor, i.e. the square root of the sum of the squares of the weights.

1973

Applied Time Series Analysis

For Managerial Forecasting

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Charles R. Nelson
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Preface

This book was written with the objective of making recent developments in applied time series analysis, particularly those due to Box and Jenkins, accessible to students in business, economics, management science, and industrial engineering at the master's level. The need for such a text became evident to me when I undertook development of a forecasting course for MBA students at the Graduate School of Business of the University of Chicago. Although univariate time series analysis offers a powerful tool for forecasting in many operational settings, and should, I felt, constitute the core of such a course, it became apparent that little of what has been written in that area may be read and understood by students lacking substantial preparation in statistics. Nevertheless, I became convinced that much of the substance of what is important for application could be communicated to a less-sophisticated audience. For example, presentation of the theory of linear stochastic processes is greatly facilitated by use of the algebra of back-shift operators. Unfortunately, some considerable investment in time, relative to that available in a single academic term, is required before most students are sufficiently comfortable and skilled in using backshift algebra to benefit from the investment. Consequently, in

this text many general results that are readily proved using backshift algebra are simply stated after their plausibility and intuitive appeal have been established by simple examples. Experimentation with this approach, combined with heavy emphasis on first-hand data analysis, led to a workable program that equips the student to make intelligent use of time series analysis and provides him with a base for further reading or formal study in the area.

Chapter 1 reviews the primary methodological alternatives open to the operational forecaster and establishes the motivation for putting particular emphasis on univariate time series analysis. The introductory discussion goes on to consider why there is a payoff to good forecasting in the context of a profit-maximizing firm and the relationship between the forecaster and the decision-maker. Chapter 2 begins the formal study of time series analysis with the concepts of stationarity and autocorrelation. Models for stationary time series are presented in Chapter 3. The moving-average model is introduced as a special case of the general discrete linear stochastic process, after which autoregressive models and mixed autoregressive-moving-average models are seen to be natural conceptual extensions. Considerable attention is given to derivation of the autocorrelation structures of these models and the observational characteristics for the data that those structures imply. Chapter 4 widens the scope of the linear models to include non-stationary behavior by entertaining the stationary models of Chapter 3 as models for the differences or successive changes occurring in nonstationary series. With this extension, the linear models are seen to offer a very flexible framework in which to describe the behavior of a wide range of stationary and nonstationary series. Chapter 5 takes up the problem of statistical inference—namely, choice of a model appropriate to a particular time series ("identification") and use of the data at hand to estimate the parameters of the model. Considerable emphasis is placed on understanding the limits of precision encountered in practical application and on interpretation of results obtained in illustrative examples. Chapter 6 completes the model-building sequence with computation of forecasts and confidence intervals and derivation of rules for adaptive revision of forecasts. Seasonality is a property of many time series of interest to the operational forecaster, and a special class of linear models is required to model such series. These are discussed in Chapter 7, where their basic stochastic properties are developed and identification and estimation are illustrated by application to monthly auto registrations in the United States. No forecasting strategy should proceed in a critical vacuum. Rather, alternative techniques should be subjected to comparison with a view toward discovering their relative strengths and weaknesses and, ultimately, revision of forecasting strategy. Chapter 8 is addressed to the question of forecaste evaluation, offering a methodological approach that emphasizes optimal weighting of alternative forecasts in composite forecasts. Evaluation of forecasts of major macroeconomic variables by the Federal Reserve Board-MIT-Penn econometric

Contents

1974

The Analysis of Time Series: Theory and Practice

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Lecturer in Statistics,
The University of Bath

CHAPMAN AND HALL
London

A HALSTED PRESS BOOK
JOHN WILEY & SON New York

Preface

Time-series analysis is an area of statistics which is of particular interest at the present time. Time series arise in many different areas, ranging from marketing to oceanography, and the analysis of such series raises many problems of both a theoretical and practical nature. I first became interested in the subject as a postgraduate student at Imperial College, when I attended a stimulating course of lectures on time-series given by Dr. (now Professor) G. M. Jenkins. The subject has fascinated me ever since.

Several books have been written on theoretical aspects of time-series analysis. The aim of this book is to provide an introduction to the subject which bridges the gap between theory and practice. The book has also been written to make what is rather a difficult subject as understandable as possible. Enough theory is given to introduce the concepts of time-series analysis and to make the book mathematically interesting. In addition, practical problems are considered so as to help the reader tackle the analysis of real data.

The book assumes a knowledge of basic probability theory and elementary statistical inference (see Appendix III). The book can be used as a text for an undergraduate or postgraduate course in time-series, or it can be used for self tuition by research workers.

Throughout the book, references are usually given to recent readily accessible books and journals rather than to the original attributive references. Wold's (1965) bibliography contains many time series references published before 1959.

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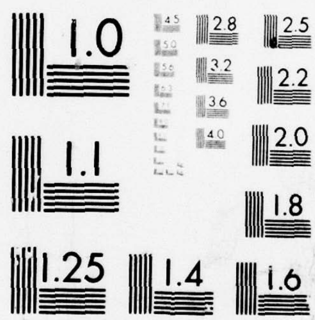

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

PREFACE

In writing the book, I have been struck by the many different definitions and notations used by different writers for different features of time-series. This makes the literature rather confusing. I have adopted what seems to me to be the most sensible definition and notation for each feature, but I have always tried to make it clear how my approach is related to that of other authors.

A difficulty with writing a textbook is that many practical problems contain at least one feature which is 'non-standard' and these cannot all be envisaged in a book of reasonable length. Thus I want to emphasize that the reader who has grasped the basic concepts of time-series analysis, should always be prepared to use his commonsense in tackling a problem. Example 1 of Section 5.5 is a typical situation where common-sense has to be applied and also stresses the fact that the first step in any time-series analysis should always be to plot the data.

I am indebted to many people for helpful comments on earlier drafts, notably Chris Theobald, Mike Pepper, John Marshall, Paul Newbold, David Prothero, Henry Neave, and Professors V. Barnett, D. R. Cox, and M. B. Priestley. Dick Fenton carried out the computing for Section 7.9. I would particularly like to thank Professor K. V. Diprose who made many useful suggestions regarding linear systems and helped me write Section 9.4.3. Of course any errors, omissions or obscurities which remain are entirely my responsibility. The author will be glad to hear from any reader who wishes to make constructive comments.

Finally it is a particular pleasure to thank Mrs. Jean Honebon for typing the manuscript with exceptional efficiency.

School of Mathematics,
University of Bath.
December 1974.

Christopher Chatfield.

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CONTENTS

1974

The Spectral Analysis of Time Series

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Preface

This book is intended to provide an introduction to the techniques and theory of the frequency domain (spectral) analysis of time series. It has been written for use both as a textbook and for individual reading by a rather diverse and varied audience of time series analysis "users." For this purpose, the style has been kept discursive and the mathematical requirements have been set at the minimum level required for a sound understanding of the theory upon which the techniques and applications rest. It is essential even for the reader interested only in the applications of time series analysis to have an understanding of the basic theory in order to be able to tailor time series models to the physical problem at hand and to follow the workings of the various techniques for processing and analyzing data. Acquiring this understanding can be a stimulating and rewarding endeavor in its own right, because the theory is rich and elegant with a strong geometric flavor. The geometric structure makes possible useful intuitive interpretations of important time series parameters as well as a unified framework for an otherwise scattered collection of seemingly isolated results. Both features are exploited extensively in the text.

The book is suitable for use as a one-semester or two-quarter course for students whose mathematical background includes calculus, linear algebra and matrices, complex variables through power series, and probability and statistics at the postcalculus level. For students with more advanced mathematical preparation, additional details and proofs of several of the results stated in the text are given in the appendices.

The basic geometry of vector spaces used throughout the book is summarized in Chapter 1 and the various (nonprobabilistic) models possessing

spectral decompositions required in later chapters are presented as applications of the geometric theory. The univariate, continuous-time models used in spectral analysis are introduced in Chapter 2 and the discrete-time models are given in Chapter 3 along with a discussion of the sampling of time series. Chapter 4 contains a general discussion of linear filters while Chapter 6 is concerned with a variety of special purpose filters in discrete time (digital filters). Multivariate time series models are introduced in Chapter 5 and a number of examples illustrating the use and interpretation of the multivariate spectral parameters are given. The standard finite parameter time series models are presented in Chapter 7 along with a discussion of linear prediction and filtering.

The statistical theory of spectral analysis is covered in Chapters 8 and 9. The distributions of spectral estimators are derived in Chapter 8 and are applied to the calculation of confidence intervals and hypothesis tests for the more important spectral parameters. The properties of spectral estimators as point estimates are considered in Chapter 9. This chapter also contains a discussion of the experimental design of spectral analyses and of the various computational methods for estimating spectra. The necessary tables for the hypothesis tests, confidence intervals, and experimental design methods covered in the text are provided in the appendix to Chapter 9.

This book contains no (formal) sets of exercises. It is my philosophy that a course in time series analysis should be tailored to the students' needs and this is best reflected in the kinds of activities required of them. In this regard, the exercises should be determined by the interests and preparation of the audience. For graduate students in mathematics and statistics, mathematical exercises will be appropriate, and several will be suggested to the instructor in the form of enlargements on the theory in the text and the appendices. Students with more applied interests should devote most of their effort to familiarizing themselves with the methods and computer programs for performing the analyses described in the text and to applying these techniques to simulated time series and to actual data from their fields of study. There is absolutely no substitute for practical experience in learning this subject. In fact, even the more theoretically oriented students of time series analysis should undertake some activities of this nature.

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1975

Time Sequence Analysis in Geophysics

E.R. Kanasewich

Second revised edition

The University of Alberta Press
1975

PREFACE

Time sequence analysis is carried out extensively by earth scientists in industry and in research laboratories. Courses on the subject are being introduced in every department that pursues geophysics. However, there is a lack of a suitable text which gathers together material published in many journals. The available published books emphasize either the purely mathematical aspect or the application to engineering, radar technology, or economics. Care has been taken to relate the subject material to the fields of physics, electrical engineering and geophysics. Original authors have been given credit for their published research discoveries and the references are included. A portion of the contents of this book has been used in a semester course during the past five years either at the senior undergraduate level or the graduate level. Additional material, from the fields of seismology and geomagnetism as applied in industry, is assigned to the students as subjects for a research project and a seminar. The lecturer will find this material readily available in journals of the Society of Exploration Geophysics or the European Society of Exploration Geophysics. This book is intended to present the fundamental background necessary for digital processing of geophysical or other types of experimental data. It is assumed that the digital computer will be used in obtaining spectral density and in the application of correlation, convolution and deconvolution techniques. Since the author has received many requests from colleagues at other universities for advance copies of the manuscript which he has been unable to supply, a photo reduction from a type-written manuscript was used for the first edition to reduce the time lag between the completion of the manuscript and the final printed copy. Much active research is still being done in this field but it is felt that the subjects presented here will assist students in reading the current literature.

The title given to this study is time sequence analysis since many of the applications involve elastic and electromagnetic waves from well defined sources and it is desired to study relations between a sequence of data points or a sequence of signals in order to determine the physical properties of the earth. Some of the processes involved in contributing to the observed data will be deterministic while others will be random. The origin time of the wave pulse is often known or is a quantity to be determined. The use of the phrase "time series" is usually reserved for a study of random events or for data whose properties are independent of translation of the origin of time.

I am greatly indebted to my colleagues for suggestions and encouragement. It is a pleasure to acknowledge the assistance of my graduate students, Messrs. R.G. Agarwal, R.M. Clowes, C.D. Hemmings and J.F. Montalbetti for checking various portions of the draft manuscript and for writing and running various computer programs. I would like to thank Mrs. Helen Hawkes for her assistance in preparing the manuscript.

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1976

FOURIER ANALYSIS OF TIME SERIES: AN INTRODUCTION

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John Wiley & Sons

New York • London • Sydney • Toronto

PREFACE

There are two groups of users of an applied book on time series analysis such as this. The first consists of students (undergraduate or postgraduate) who encounter a course on time series in their study of statistics or its allied fields. The second consists of workers in the many fields in which time series data arise. The fact that this book was written with both groups in mind has imposed noticeable constraints on the contents and presentation, but they have proved entirely beneficial.

In the interests of the second group, the statistical level of the presentation has been kept low. The minimum statistical knowledge needed to follow the essential sections corresponds to a single introductory course in statistics. Greater knowledge of statistics, combined with some experience in the analysis of observational data, would of course allow the reader both an easier passage and the opportunity for greater gain on the way.

The interests of students are best served, at least on their first contact with time series, by tying the presentation to examples. All the methods described in this book are introduced in the context of specific sets of data, so that the motivation behind a method is evident as it is developed. The abstract properties of a procedure are discussed only when the motivation has been solidly established.

Many people have difficulty when they first encounter Fourier analysis or the Fourier transform. The discrete Fourier transform is described in Chapter 3 and is used in one form or another through most of the remaining chapters. It is elementary from a mathematical point of view, involving nothing more advanced than the summation of finite series, not even calculus. However its properties are analogous to those of more difficult types of Fourier transforms. Careful study of Chapter 3 and the expenditure of some time on its exercises will convince the most faint-hearted that Fourier transforms can be fun! Complex numbers are used extensively in deriving the properties of the discrete Fourier transform.

However this is done purely for the notational simplicity, and to follow the algebra it is necessary only to know the rules for obtaining the sum and the product of two complex numbers.

The topics discussed are as follows:

(i) harmonic regression: least squares regression on a sinusoid or sinusoids (Chapter 2);
(ii) harmonic analysis: the discrete Fourier transform, periodogram analysis (Chapters 3, 4, and 5);
(iii) complex demodulation (Chapter 6);
(iv) spectrum analysis (Chapters 7, 8, and 9).

The order of the discussion is dictated by the increasing complexity of the statistical concepts involved. At all stages of the book, the reader is urged to stop and consider the appropriateness of applying a particular method to the set of data under consideration. In several cases some preprocessing is carried out to make the data more appropriate. This is all designed to make the point that any data-analytic procedure based on the sine and cosine functions has a better chance of yielding useful conclusions if the data show some kind of oscillation, preferably as uniform or as regular as possible.

There are exercises at the ends of most sections. Some are algebraic manipulations designed to make the reader more familiar with the tools of discrete Fourier analysis and to build his or her confidence. The others are used to indicate some of the directions in which the theory of time series analysis has revealed useful results. However, the most important exercise, and one that should be omitted by no serious reader, is the analysis of data. Almost all the data used in this book are widely available. However the reader who tries the methods on data arising in his or her own field will gain the added benefit of seeing these data from a new point of view. Many of the more general purpose computer programs used to analyze the examples have been included in Appendices to the relevant chapters. They are coded in a moderately transportable dialect of FORTRAN (apart from the use of the symbol ≠ rather than ' to delimit character strings in FORMAT statements), and all have been executed successfully. Naturally they are not guaranteed to be bug-free.

I was encouraged to write this book by Geoffrey Watson, who saw clearly the need for an introductory text on Fourier methods not encumbered by an abundance of mathematical, probabilistic, or statistical detail. An early version was used as class notes in an undergraduate course on time series taught in the Department of Statistics at Princeton University in the spring of 1974. Revision was begun during a visit to the Computer Centre and the Department of Statistics, Institute of Advanced Study, Australian National University, Canberra. The final draft was

prepared during a leave spent at the Department of Statistics, University of California at Berkeley. I am grateful to colleagues at all three institutions for their assistance and encouragement, especially David Brillinger, Richard Hamming, E. J. Hannan, and John Tukey. Each institution provided an excellent background within which to work, including library and computer facilities.

The plasma physics data used in Chapters 6 and 9 were kindly provided by Joseph Cecchi of the Plasma Physics Laboratory, Department of Astrophysics, Princeton University. I also thank Michael Stoto and John Turner for their assistance with computing problems. The final draft was typed immaculately, with great good humor, by Ruth Suzuki. Much of the computing was supported by the Office of Naval Research, under contract number 0014-67-A-0151-0017 with the Department of Statistics, Princeton University.

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1976

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Fundamentals of Geophysical Data Processing

WITH APPLICATIONS TO PETROLEUM PROSPECTING

PREFACE

This book is based at the level of a bachelor's degree in physical science. Experience at Stanford indicates that a one-semester class in engineering systems theory provides helpful additional background. It will be readable to a general science and engineering audience and should be useful to anyone interested in computer modeling and data analysis in physical sciences. Inevitably, the book is strongly flavored by my own research interests which are presently mainly in exploration seismology. However, I have taken an interest in a good many of the data processing problems in general geophysics which have arisen in eight years of teaching graduate students and supervising research. This book is intended to be a textbook rather than a research monograph. The exercises are of a reasonable degree of difficulty for first-year graduate students, and most of them have been thoroughly tested.

I am indebted to a great many friends, associates, and former teachers for much of what I have learned. I have had many fruitful conversations with Steve Simpson, Enders Robinson, and John Burg about time series analysis. Ted Madden taught me much of what is written in this book on stratified media, but most importantly he infected me with the idea that the time had come to go beyond stratified media. John Sherwood and Francis Muir introduced me to reflection

seismic prospecting and some unorthodox ways of thinking about it. Several generations of students were a great help in getting many of the "bugs" out of the text and the exercises. Phil Schultz, Don C. Riley and Steve Doherty prepared many of the figures in the final chapters. Mrs. Susana Erlin typed most of the manuscript and finally got the effort all together. My wife, Diane, inspired the continuing effort the project required.

Thanks for financial support over the past eight years is due mainly to Stanford University and the Chevron Oil Field Research Company, but also to the Petroleum Research Fund of the American Chemical Society, the National Science Foundation, and the Air Force Office of Scientific Research. Recent support from the sponsors of the Stanford Exploration Project (SEP) has enabled the rapid development of wave equation seismic data processing introduced in the last chapter. These sponsors are: Amoco, Arco, Chevron, Continental, Digicon, Dutch Shell, ELF-France, Exxon, GSI, INA-Yugoslavia, Mobil, Petrofina-Belgium, Petty Ray, Preussag-Germany, Seiscom Delta, Seismograph Service, Shell, Sun, Teledyne, Texaco, Total-France, Union, U.S. Geological Survey, United Geophysical, and Western Geophysical.

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1976

INTRODUCTION TO STATISTICAL TIME SERIES

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New York • London • Sydney • Toronto

Preface

This textbook was developed from a course in time series given at Iowa State University. The classes were composed primarily of graduate students in economics and statistics. Prerequisites for the course were an introductory graduate course in the theory of statistics and a course in linear regression analysis. Since the students entering the course had varied backgrounds, chapters containing elementary results in Fourier analysis and large sample statistics, as well as a section on difference equations, were included in the presentation.

The theorem-proof format was followed because it offered a convenient method of organizing the material. No attempt was made to present the most general results available. Instead, the objective was to give results with practical content whose proofs were generally consistent with the prerequisites. Since many of the statistics students had completed advanced courses, a few theorems were presented at a level of mathematical sophistication beyond the prerequisites. Homework requiring application of the statistical methods was an integral part of the course.

By emphasizing the relationship of the techniques to regression analysis and using data sets of moderate size, most of the homework problems can be worked with any of a number of statistical packages. One such package is SAS (Statistical Analysis System, available through the Institute of Statistics, North Carolina State University). SAS contains a segment for periodogram computations that is particularly suited to this text. The system also contains a segment for regression with time series errors compatible with the presentation in Chapter 9. Another package is available from International Mathematical and Statistical Library, Inc.; this package has a chapter on time series programs.

There is some flexibility in the order in which the material can be covered. For example, the major portions of Chapters 1, 2, 5, 6, 8, and 9 can be treated in that order with little difficulty. Portions of the later chapters deal with spectral matters, but these are not central to the development of those chapters. The discussion of multivariate time series is positioned in separate sections so that it may be introduced at any point.

I thank A. R. Gallant for the proofs of several theorems and for the repair of others; J. J. Goebel for a careful reading of the manuscript that led to numerous substantive improvements and the removal of uncounted mistakes; and D. A. Dickey, M. Hidiroglou, R. J. Klemm, and G. H. K. Wang for computing examples and for proofreading. G. E. Battese, R. L. Carter, K. R. Crouse, J. D. Cryer, D. P. Hasza, J. D. Jobson, B. Macpherson, J. Mellon, D. A. Pierce and K. N. Wolter also read portions of the manuscript. I also thank my colleagues, R. Groeneveld, D. Isaacson, and O. Kempthorne, for useful comments and discussions. I am indebted to a seminar conducted by Marc Nerlove at Stanford University for the organization of some of the material on Fourier analysis and spectral theory. A portion of the research was supported by joint statistical agreements with the U. S. Bureau of the Census.

I thank Margaret Nichols for the repeated typings required to bring the manuscript to final form and Avonelle Jacobson for transforming much of the original illegible draft into typescript.

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Ames, Iowa
February 1976

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1976

Statistical Forecasting

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Preface

From the dawn of recorded history, and probably before, man has sought to forecast the future. Indeed, the ability to foresee the consequences of actions and events is one of the defining properties of 'mind'. Where the phenomenon being forecast was a purely physical one, such as the occurrence of midsummer's day or of an eclipse, man was able from very early times to obtain very accurate forecasts. Initially such forecasts were derived on a purely empirical basis. Methods were found that worked without any basic understanding as to why they worked. Later, as greater understanding of the phenomena developed theoretical models were developed which enabled forecasts to be obtained on a more reasoned basis. During the last century interest focused on a number of phenomena, such as economic variation and sunspot cycles, where

(a) there were available series of observations taken over a period of time, called time-series, upon which forecasts could be based;
(b) purely mathematical rules were found to be inadequate to describe the phenomena since they involved features of a chance nature.

Over the last fifty years approaches to this type of problem have been developed which seek to allow for the influence of chance and for the unknown complexity of these situations. Thus, forecasting methods were developed which were essentially statistical in nature. These were based on using statistical techniques to fit models of a wide variety of types to past time-series data and on forecasting the future from these models.

The aim of this book is to provide the reader with an understanding of the methods and practice of statistical forecasting. The structure of the book is based on the structure of the activities involved in the practice of statistical forecasting. Figure P.1 summarizes this structure. Part I of the book provides a general introduction to some of the basic concepts and terms in statistical forecasting. Part II deals with the development of statistical forecasting methods applicable to each of a variety of different situations. The approach to classifying this part into chapters has been to classify the situations; e.g. one chapter considers

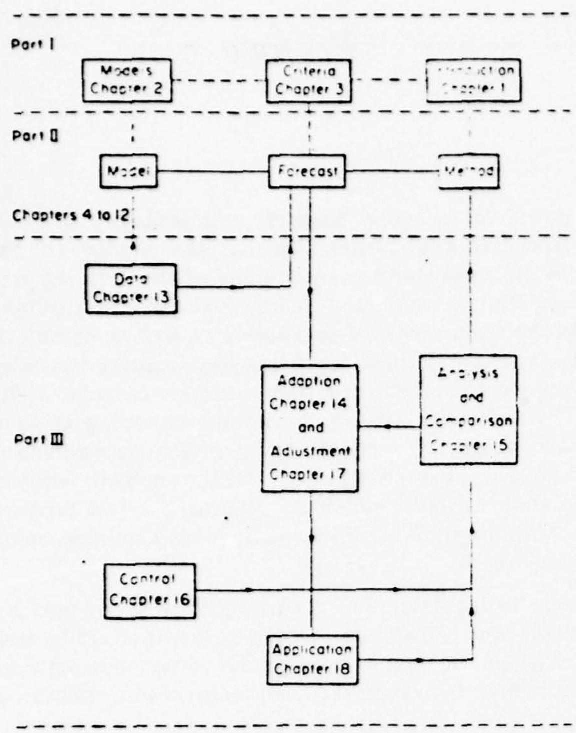

Figure P.1 The structure of the book and of statistical forecasting

trending time series, another seasonally varying time series. We then consider the various methods that might be used in each situation. In practice there is considerably more to statistical forecasting than putting data into a formula to obtain a forecast. Part III deals with these further facets. In Chapter 13 we look at the data that is needed for forecasting, where it might come from and how it may be treated. Chapter 14 looks at ways in which the forecasting methods of Part II may be adapted to apply to a wider range of applications and to give a better quality of forecast. In Chapter 15 we examine a wide range of methods for analysing and comparing forecasting methods. Chapter 16 has the same basic aim, but is conerned more with the routine control of an operational forecasting system. In Chapter 17 we use some of the properties of forecasts, that might be identified using the methods of Chapters 15 and 16, to create even better forecasts. Always we seek to bear in mind that forecasts are produced for practical purposes that may influence the way we obtain and use our forecast in the first place.

This interrelation between forecast and application is explored in Chapter 18.

The general aim of the book is to give a systematic account of the methods of statistical forecasting in common use and of the practical aspects of their use; consequently, the level of mathematics and statistics assumed of the reader has been kept to that which the writer's teaching experience suggests as appropriate to people whose interest is in obtaining forecasts for practical purposes. This is mainly limited to the basic ideas of statistics and the ability to follow simple algebraic arguments. Where more than this is used, the section is marked with an * and can be omitted without detriment to the understanding of later sections. Where certain ideas are used that might be unfamiliar to a significant proportion of readers, these are discussed in the appendices. Wherever possible, the discussion of each forecasting method is developed and illustrated by example in such a way that the reader may then be able to use the method himself. There are, however, a number of methods whose mathematical development has been regarded as beyond the scope of this book, but computer programs are commonly available for their use. The author's aim in these cases has therefore been to discuss and illustrate the basic concepts of the methods. Thus the reader should be able to use such computer programs with understanding of their principles though not of their technical details.

A short list of relevant references which are referred to in the text are given at the end of each chapter.

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Contents

PART III THE FORECASTING PROCESS

1976

Dynamic Stochastic Models from Empirical Data

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Preface

The construction of stochastic dynamic models from empirical time series is practiced in a variety of disciplines, including engineering, ecology, and applied statistics, with specific forecasting aims. However, there have been few systematic expositions of the major problems facing model builders: determination of the plausible classes of models for the given time series by inspection of the series and examination of its characteristics, detailed comparison of the various classes of models, the role and methods of model validation, etc. Of course, there have been a number of books discussing some techniques for developing models for time series data, as mentioned in the text.

The central problems in model building are, in our view, the choice of the appropriate class of models and the validation or checking for adequacy of the best fitting models from the selected class. Even though optimal parameter estimation methods are used, the performance of the best fitting model from an inappropriate class of models may be poor in comparison with the performance of the corresponding member of the appropriate class. Moreover, detailed validation tests bring out the limitations of the selected class and may suggest a more appropriate class, if one exists. In earlier expositions of the subject, the class comparison and validation problem, if considered at all, was discussed entirely in terms of the classical theory of hypothesis testing and other similar decision theoretic methods. The comparison of many important classes of models can be demonstrated to be beyond the scope of the theory of hypothesis testing because of the difficulty in finding the probability distribution of the test statistic. Furthermore, a validation program in which only the residuals are tested by using the methods of hypothesis testing is often inconclusive. Thus the development of various approaches to comparison of different classes and subsequent validation of the final models are the major themes of this book. In addition, relatively standard topics such as parameter estimation methods and estimability are covered in some detail.

The validity of the methodology developed in the text is demonstrated by presenting detailed case studies of model development for about 15 univariate and multivariate time series. In these case studies, all the important numerical details of parameter estimation, class selection, and validation are included. The rainfall and riverflow series, the animal population series, the U.S. population series, and sales figures of a certain company are some of the data sets treated here. The potential application of the model for forecasting, generation of synthetic data, and verification of certain causal hypotheses about environmental processes is discussed at some length. In particular, stochastic models are

demonstrated to be superior to deterministic models even though the latter are popular.

The plan of the book is as follows: Chapter I is an introduction. Chapter II contains a brief discussion of a number of topics, such as prediction and other prerequisite materials that will be needed in later chapters. Chapter III is a qualitative discussion of the dominant features of time series obeying various schema for models, including autoregressive moving average (ARMA) models, integrated ARMA models, covariance *stationary* models, etc., with suggested guidelines for choice of the possible classes of models for any given series. Chapters IV and V deal with the problem of model multiplicity and estimability, i.e., the conditions needed on a class of models to ensure that there is at most one model in the class for the given series. Chapters VI and VII deal with various parameter estimation methods and the corresponding tradeoff involved between estimation accuracy and computational complexity. Chapters VIII and IX deal with methods of comparison of the various classes of models and with validation *of the chosen model*. Case studies of modeling are discussed in Chapters X and XI.

This book, developed from our teaching and research at Purdue University for the past four years on the *topic of system identification*, has been designed to be used by practicing engineers, ecologists, and applied statisticians interested in constructing models and by graduate students as a textbook in a course on time series analysis or system identification. The readers of the book are assumed to have some knowledge, albeit elementary, of statistics and random processes; otherwise it is self-contained. Some of the more complicated derivations are postponed to appendixes so that the text reads smoothly. This book has been used as a textbook for a one-semester first-year graduate course with Chapter V and parts of Chapter VII omitted.

In writing a book on a subject that is being actively investigated, we can perforce describe only a few of the various methods proposed for estimation and model comparison. Our guideline in this selection has been the availability of empirical support of the methods. It is entirely possible that a number of methods that may have been successful in practice are not included here. We take comfort from the Bhagavad Gita, "All actions are tainted with some blemish."

Contents

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Some of the important references to the material used in the book are given here, although this list is not intended to be exhaustive. Wherever appropriate, we give a book or a review as a reference instead of the original paper so that additional references can be obtained from it.

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SYSTEM IDENTIFICATION

Advances and Case Studies

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PREFACE

The field of system identification and time series analysis is currently in a state of rapid development. Significant contributions have been made in the past few years by researchers from such diverse fields as statistics, control theory, system theory, econometrics, and information theory. The specialized jargon of each field, geographic isolation of researchers, and the difficulty of working on what Wiener called "cracks between disciplines" has hampered a rich cross fertilization of ideas among different specialties. The purpose of this book is to promote this activity by presenting in one volume promising new approaches and results in the field of system identification, approaches and results that are not easily available elsewhere.

The idea of putting together the current volume originated from this editor's experience with a special issue of the IEEE Transactions on Automatic Control (December 1974).* The limitations on the length of the journal papers made it very difficult for authors to expand fully on their ideas. Furthermore, significant new developments took place, which deserved widespread exposure. The effort turned out to be truly international in character with contributions from seven different countries. The authors were invited to write chapters on their current fields of interest, making their presentations self-contained and summarizing the state-of-art in their subject areas. To achieve depth and completeness in their presentations, the authors have assumed on the part of readers a basic background in statistical estimation and time series analysis, equivalent to that contained in texts such as Jenkins and Watts,[1] Box and Jenkins,[2] Graupe,[3] Sage and Melsa,[4] Eykhoff,[5] Schweppe,[6] and Astrom.[7]

Following Box and Jenkins,[2] the four steps in system identification are shown schematically in Fig. 1. The chapters in this book are organized accordingly under the following headings: (1) model structure determination, (2) parameter estimation, (3) experimental design, (4) special topics, and (5) case studies.

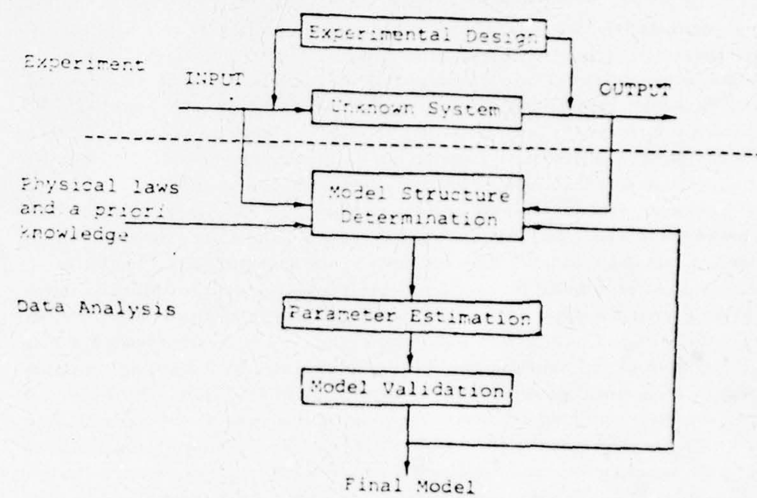

Fig. 1. Steps in System Identification

A brief description of each chapter is given below.

In Chapter I, Faurre introduces the linear Markovian representation of a time series and discusses the problem of obtaining a whole class of representations from the covariance function. He points out the importance of two special Markovian representations, one of which corresponds to the minimum variance Kalman filter for the process. Akaike, in Chapter II, further expands on this representation and discusses in detail his elegant method for determining the structure of this representation from noisy input-output data. For model order determination, Akaike uses an information criterion and illustrates his method with a number of interesting examples. Akaike's procedure is easy to implement and constitutes a major contribution to the analysis of multiple time series. It is interesting to note that a solution to this long-standing problem in time series analysis requires use of concepts from modern control and system theory, such as canonical forms and state vector models. Chapter III by Rissanen develops a new criterion for model structure determination based on the information-theoretic concept of entropy. These concepts are likely to play an increasingly important role in future developments of the system identification.

Chapters IV and V by Ljung and Narendra respectively consider the problem of consistent and stable estimation of parameters in adaptive closed loop systems. Ljung presents new methods for proving consistency and shows that the prediction error minimization method is consistent under very weak conditions. Narendra discusses on-line estimation of parameters using a model reference approach and Lyapunov's direct method. The effectiveness of this method is demonstrated by numerous examples and extensive simulation results.

Chapters VI and VII by Mehra and Goodwin and Payne respectively present new results on the choice of inputs and sampling rates. In practice, the success of system identification is often dependent on these two factors, which are generally chosen on an ad hoc basis for convenience in experimentation. A study of these two chapters reveals that methods are now available for computing both optimal and good suboptimal experimental designs for system identification.

The special topics discussed in Chapters VIII, IX, and X by Attasi, Caines and Chan, and Robinson respectively pertain to the identification and estimation of doubly indexed time series (or random fields), feedback systems and continuous time systems. Attasi presents a new state vector model for discrete random fields, such as those encountered in image processing and gravity modeling, and develops a complete theory of stochastic realization and recursive estimation for these models. The parallels between his theory and that discussed by Faurre in Chapter I are remarkable considering the fact that causality does not hold in the case considered by Attasi. A special feature of Attasi's model for random fields is that vector white noise inputs are used to obtain a recursive structure for the model, and for the statistical smoother. In Chapter IX, Caines and Chan present a thorough rigorous discussion of feedback and the identification of closed-loop systems. They also present results from applications in the areas of economics, power systems, and physiology. In Chapter X, Robinson discusses the important problem of identifying a continuous time model using discrete or sampled data. He considers the effect of "aliasing" on the cross-spectral method for obtaining both parametrized and unparametrized models for multiple time series. Robinson's chapter provides a very good balance to the rest of the book in that it contains clear exposition of the spectral methods, which do not receive their full share of attention in the other chapters.

The last two chapters of the book are devoted to case studies. Bohlin (Chapter XI) presents four case studies relating to dryer control in a paper mill, EEG signals with changing spectra, machine failure forecasting, and load forecasting in power systems. A unified procedure based on Gauss-Markov models for changing system parameters, Kalman filtering, and maximum likelihood estimation is used successfully in all four applications. The chapter contains important insights that the author has gained over the years through extensive experience with real data. In Chapter XII, Olsson presents a detailed and careful study relating to the modeling and identification of a nuclear reactor, a problem that is of great current interest for safety reasons. The chapter serves as a good example of the way a practical system has to be studied using different methods. The application of different techniques for system identification is not a luxury but a necessity when one is dealing with complex real-life systems that never fit neatly into any standard theoretical mold. Each technique properly applied gives some insight into the system and helps to reinforce the results obtained from other techniques.

The references at the end of each chapter constitute an extensive bibliography on the subject of system identification.

This volume would not have been possible without the full and dedicated participation of all the authors, to whom the editors are highly indebted. Special thanks are due to Mrs. Renate D'Arcangelo for typing most of the book in such a short period of time with partial help from Karin Young. The international scale of the effort required special coordination skills for which thanks are due to Marie Cedrone.

Finally, I would like to thank my wife, Anjoo, for her patience and understanding during long hours of work in preparing this volume.

Raman K. Mehra

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Forecasting and Time Series Analysis

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Preface

Statistical forecasting techniques are widely used in the management of production and inventory systems and have also found frequent application in a variety of other problem areas, including quality and process control, financial planning, marketing, investment analysis, and distribution planning. Despite the wide application of statistical forecasting techniques, there is not a text available that covers the range of short-term forecasting methods in an introductory fashion. We believe that this book serves that purpose and is suitable for both undergraduate students and professional practitioners who design forecasting systems.

This text has evolved from forecasting lectures in an undergraduate course in production and from a graduate course in forecasting at the Georgia Institute of Technology. We also benefited from experience in extension and continuing education activity and professional consulting in forecasting and production control.

The book can be used by readers with modest mathematical and statistical training, provided they skip some developments and take the associated results on faith. A reader familiar with calculus and introductory statistics can read the entire book. Certain sections and some chapters which have considerable mathematical content and which may be skipped without loss of continuity have been marked with an asterisk.

This book can be used in several ways. It contains exercises and examples and can serve as a text for a one-semester or one-quarter course or seminar on forecasting, such as is typically taught by departments of industrial engineering, management science, operations research, or business administration. Most of the widely used forecasting techniques are organized and presented in a manner that should make the book useful to professional practitioners who are developing and maintaining forecasting systems. The book also contains computer programs for several of the forecasting methods discussed.

The scope of the book is confined to short-term forecasting methods. Chapter 1 is an introductory discussion of the forecasting problem and of the methods and systems in general use today. This chapter also introduces terminology and notation used in the rest of the text. Chapter 2 discusses regression methods and introduces the moving average as a forecasting technique for certain simple time series structures. In Chap. 3 exponential smoothing methods are introduced. Single and double smoothing are presented, as well as the generalization to smoothing of order k for a polynomial of degree k-1. Direct smoothing of the coefficients in polynomial and transcendental models is described in Chap. 4. This chapter requires some knowledge of matrix algebra, and may be skipped by the reader. Chapter 5 presents both exponential smoothing methods for forecasting seasonal time series. Here results from Chap. 4 are used to develop efficient parameter updating procedures for trigonometric seasonal models. Chapter 6 discusses forecasting with time series models, with emphasis on the construction of prediction intervals. Methods for directly forecasting the percentiles of the probability distribution of the process are also described. The analysis of forecast errors and the use of tracking signals to monitor forecasting system performance are discussed in Chap. 7. Chapter 8 contains several procedures for automatic control of the smoothing constant. Chapter 9 presents the Box-Jenkins approach of time series modeling and forecasting. This chapter requires a more advanced statistical background than does the rest of the book. Finally, in Chap. 10 we discuss Bayesian methods for forecasting when little or no historical data are available.

Many individuals contributed to the completion of this book. We particularly thank L. E. Contreras and D. H. Vatz for their assistance in developing the computer programs in Appendix C, B. W. Schmeiser for providing Table A-4, and Professor J. A. White for supplying the data in Example 9-5. We are also indebted to Professor E. S. Pearson and the Biometrika Trustees, the editor of *AIIE Transactions*, the editor of *Operationl Research Quarterly*, and The Ronald Press Company for permission to use copyrighted material. We thank Dr. R. N. Lehrer for providing resources in support of this project. Finally, we thank the several secretaries involved in typing this manuscript.

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ECONOMETRIC MODELS AND ECONOMIC FORECASTS

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INTRODUCTION

This book is an introduction to the science and art of building and using models. The science of model building consists of a set of tools, most of them quantitative, which are used to construct and then test mathematical representations of portions of the real world. The development and use of these tools are subsumed under the subject heading of econometrics. While not altogether straightforward, econometrics is a well-defined field and therefore relatively easy to describe in written prose. For this reason, the science of model building will be a primary concern of this book. The art of model building is, unfortunately, much harder to describe in words, since it consists mostly of intuitive judgments that occur during the modeling process. Unfortunately, there are no clear-cut rules as to how these model-building judgments should be made, which makes the art of model building difficult to master. Nonetheless, one of the purposes of this book is to convey the nature of that art to the reader to the extent possible. This will be done in part by examples and by discussions of technique, but also by encouraging the reader to do what is ultimately necessary to master the art, namely, to build models of his her own.

The book focuses upon models of processes that occur in business, economics, and the social sciences in general. These might include models of aggregate economic activity, the sales of an individual firm, or a political process (e.g., estimating the number of votes that a particular candidate can be expected to receive in an election). Discussions of the purposes of building these models are directed toward forecasting and policy analysis, but the reader should bear in mind the general nature of the content.

As one might expect, there are many types of models that can and often have been used for policy analysis and forecasting. This book does not attempt to cover the entire spectrum of model types and modeling methodologies; rather, it concentrates on those models which fit (and test theoretical relationships against) data. This still leaves a rather wide range of models among which to choose. On one end of this range, one might determine the effect of alternative monetary policies on the behavior of the United States economy by constructing a large, multi-equation econometric model of the economy and then simulating it using different monetary policies. The resulting model would be rather complicated and would presume to explain a complex structure in the real world. On the other end of the

range, one might desire to forecast the sales volume of a firm and, believing that those sales follow a strong cyclical pattern, use a time-series model to extrapolate the past behavior of sales.

It is this range of models that is the subject matter of this book. The objective of this book is to give the reader some understanding of the science and art of determining what type of model to build, building the model which is most appropriate, testing the model statistically, and then finally applying the model to practical problems in forecasting and analysis.

1 WHY MODELS?

Many of us often either use or produce forecasts of one sort or another. Few of us recognize, however, that some kind of logical structure, or model, is implicit in every forecast or analysis of a social or a physical system. Consider, for example, a stockbroker who tells you that the Dow Jones Industrial Average will rise next year. The stockbroker might have made this forecast because he/she has seen the Dow Jones average rising during the past few years and feels that whatever it was that made it rise in the past will continue to make it rise in the future. On the other hand, he/she might feel that the Dow Jones Industrial Average will rise next year because of a belief that this variable is linked to a set of economic and political variables through a complex set of relationships. The broker might believe, for example, that the Dow Jones average is related in a certain way to the gross national product and to interest rates, so that given certain other beliefs about the most probable future behavior of these latter variables, he/she would be led to believe that a rise in the Dow Jones average is likely.

If we have to find a word to describe the method by which our stockbroker made this forecast, we would probably say that it was intuitive, although the chain of reasoning differed substantially in the two cases cited above. The stockbroker certainty would not say that the forecast was made by building a model of the Dow Jones average; indeed, no equations were written down, nor was any computer used. Nonetheless, in each case, some *implicit* form of model building was involved. If the stockbroker based the optimistic forecast for the Dow Jones average on past increases, he/she has in effect constructed a *time-series model* which extrapolates past trends into the future. If, instead, the forecast was based on a knowledge of economics, a model would still be implicitly involved—it would be composed of the relationships that were loosely conceived in the stockbroker's mind as a result of past experience.

Thus, even the intuitive forecaster constructs some type of model, even if he/she is not aware of doing so. Of course, it is reasonable to ask why one might want to work with an *explicit* model to produce forecasts. Would it be worth the trouble, for example, for our stockbroker to read this book in order to construct an explicit model, estimate it on the computer, and test it statistically? Our response is that there are several advantages to working with models explicitly. Model building forces the individual to think clearly about and account for all the important interrelationships involved in a problem. The reliance on intuition can be dangerous

at times because of the possibility that important relationships will be ignored or improperly used. In addition, it is important that individual relationships be tested or validated in some way or another. Unfortunately, this is not usually done when intuitive forecasts are made. In the process of building a model, however, the individual must test or validate not only the model as a whole, but also the individual relationships that make up the model.

When making a forecast, it is also important to provide a statistical measure of confidence to the user of the forecast, i.e., some measure of how accurate one might expect the forecast to be. The use of purely intuitive methods usually precludes any quantitative measure of confidence in the resulting forecast. The statistical analysis of the individual relationships that make up a model, and of the model as a whole, makes it possible to attach a measure of confidence to the model's forecasts.

Once a model has been constructed and fitted to data, a sensitivity analysis can be used to study many of its properties. In particular, the effects of small changes in individual variables in the model can be evaluated. For example, in the case of a model that describes and predicts interest rates, one could measure the effect on a particular interest rate of a change in the rate of inflation. This type of quantitative sensitivity study, which is important both in understanding and in using the model, can be done only if the model is an explicit one.

2 TYPES OF MODELS

In this book we examine three general classes of models that can be constructed for purposes of forecasting or policy analysis. Each involves a different degree of model complexity and structural explanation, and each presumes a different level of comprehension about the real world processes that one is trying to model. The three classes of models are as follows:

(a) *Time-Series Models* In this class of models we presume to know nothing about the real world causal relationships that affect the variable we are trying to forecast. Instead we examine the past behavior of a time series in order to infer something about its future behavior. The time-series method used to produce a forecast might involve the use of a simple deterministic model such as a linear extrapolation, or the use of a complex stochastic model for adaptive forecasting.

One example of the use of time-series analysis would be the simple extrapolation of a past trend in predicting population growth. Another example would be the development of a complex linear stochastic model for passenger loads on an airline. Models such as this have been developed and used to forecast the demand for airline capacity, seasonal telephone demand, the movement of short-term interest rates, as well as other economic variables. Time-series models are particularly useful when little is known about the underlying process that one is trying to forecast. The limited structure in time-series models makes them most reliable only in the short run, but they are nonetheless rather useful.

(b) *Single-Equation Regression Models* In this class of models the variable under

study is explained by a single function (linear or nonlinear) of explanatory variables. The equation will often be time-dependent (i.e., the time index will appear explicitly in the model), so that one can predict the response over time of the variable under study to changes in one or more of the explanatory variables.

An example of a single-equation regression model might be an equation that relates a particular interest rate, such as the 3-month Treasury bill rate, to a set of explanatory variables such as the money supply, the rate of inflation, and the rate of change in the gross national product. Regression models are often used to forecast not only the movement in short- and long-term interest rates, but also many other economic and business variables.

(c) *Multi-Equation Simulation Models* In this class of models the variable to be studied may be a function of several explanatory variables, but now these explanatory variables are related to each other as well as to the variable under study through a set of equations. The construction of a simulation model begins with the specification of a set of individual relationships, each of which is fitted to available data. Simulation is the process of solving these equations simultaneously over some range in time.

An example of a multi-equation simulation model would be a complete model of the United States textile industry that contains equations that explain variables such as textile demand, textile production output, employment of production workers in the textile industry, investment in the industry, and textile prices. These variables would be related to each other and to other variables (such as total national income, the Consumer Price Index, interest rates, etc.), through a set of linear or nonlinear equations. Given assumptions about the future behavior of national income, interest rates, etc., one could simulate this model into the future and obtain a forecast for each of the model's variables. A model such as this can be used to analyze the impact on the industry of changes in external economic variables.

Multi-equation simulation models presume to explain a great deal about the structure of the physical process that is being studied. Not only are individual relationships specified, but the model accounts for *the interaction of all these interrelationships* at the same time. Thus, a five-equation simulation model actually contains more information than the sum of five individual regression equations. The model not only explains the five individual relationships, but it also describes the dynamic structure implied by the simultaneous operation of these relationships.

The choice of the type of model to develop is a difficult one, involving trade-offs among time, energy, costs, and desired forecast precision. The construction of a multi-equation simulation model might require large expenditures of time and money, not only in terms of actual work, but also in terms of computer time. The gains that result from this effort might include the better understanding of the relationships and structure involved as well as the ability to make a better forecast. However, in some cases these gains may be small enough so that they are outweighed by the heavy costs involved. Because the multi-equation model necessitates

a good deal of knowledge about the process being studied, the construction of such models may be extremely difficult.

The decision to build a time-series model usually occurs in cases when little or nothing is known about the determinants of the variable being studied, when a large number of data points are available (thus making some kind of inference feasible), and when the model is to be used largely for short-term forecasting. Given some information about the processes involved, however, it may not be obvious whether a time-series model or a single-equation regression model is preferable as a means of forecasting. It may be reasonable for a forecaster to construct both types of models and compare their relative performances.

In the course of this book, we plan not only to describe how each type of model is constructed and used, but also to give some insight into the relative costs and benefits involved. Unfortunately, this can be a rather hard problem, as the choice of model type is often not clear. In any case, it seems natural to include a discussion of all three types of models (single-equation regression, multi-equation simulation, and time series) in the confines of a single book.

3 WHAT THE BOOK CONTAINS

The book is divided into three parts, each concentrating on a different class of models. *The most fundamental class of models, discussed in the first part of the book,* is the single-equation regression model. The econometric methods developed and used to construct single-equation regression models will, with modification, find application in the construction of multi-equation simulation models as well as time-series models. Thus, Part One of this book presents an introduction to the development and estimation of single-equation econometric models.

Chapter 1 begins with an introduction to the basic concepts of regression analysis. The regression model then is developed in detail, beginning with a two-variable model in Chapter 2 and proceeding to the multiple regression model in Chapter 3. These chapters also develop statistical tests and procedures that can be used to evaluate the properties of a regression model.

The estimation techniques used in simple regression analysis require that certain assumptions be made about both the data and the model. At times, these assumptions break down. Chapters 4 and 5 begin a discussion of what can be done in some of these cases. Chapter 4 deals with heteroscedasticity and serial correlation and includes statistical tests for these problems as well as estimation methods that correct for them. Chapter 5 deals with the problem of correlation between explanatory variables and the implicit error term in the regression model. It concentrates on the development of the instrumental variable and two-stage least-squares estimation techniques.

Chapter 6 discusses the use of a single-equation regression model for forecasting purposes. The chapter discusses not only the methods by which a forecast is produced, but also measures that describe the reliability of a forecast, such as confidence intervals and the error of forecast.

The last two chapters of Part One of the book consider extensions of the regres-

sion model. These chapters are somewhat more advanced in nature, and could be skipped by the beginning student. Chapter 7 deals with the problems of specification error, missing observations, the estimation of nonlinear models, distributed lag models, and models which pool cross-section and time-series data. Chapter 8 deals with models in which the variable to be explained is qualitative in nature. These include linear probability models, probit models, and logit models.

The foundation of econometrics of Part One is essential for the development of multi-equation simulation models in Part Two of the book. Part Two begins with a chapter on estimation techniques particular to simultaneous equation models. This includes problems of model identification, as well as techniques such as three-stage least squares. Chapters 10 and 11 discuss the methodology of constructing and using multi-equation simulation models. Chapter 10 is an introduction to simulation models, and includes a discussion of the simulation process, methods of evaluating simulation models, alternative methods of estimating simulation models, and general approaches to model construction. Chapter 11 is more technical in nature and discusses methods of analyzing the dynamic behavior of simulation models, including questions of model stability, dynamic multipliers, and methods of tuning and adjusting simulation models. Chapter 11 concludes with a discussion of sensitivity analysis and stochastic simulation.

Part Two closes with a chapter that presents three detailed examples of the construction and use of simulation models. In the first example, a small but complete model of the United States economy is constructed and used for simple policy analysis. The second example develops an industry market model and shows how it can be used to forecast production and prices. The last example shows how simulation techniques can be useful for financial planning in a corporation.

Part Three of this book is devoted to time-series models, which can be viewed as a special class of single-equation regression models. Thus, the econometric tools developed in Part One of the book will find extensive application in Part Three. Part Three begins with Chapter 13, which introduces the basic properties of random time series, as well as the notion of a time-series model. The chapter discusses the properties of stationary and nonstationary time series, and the calculation and use of the autocorrelation function.

Chapters 14, 15, and 16 develop the methods by which time-series models are specified, estimated, and used for forecasting. Chapter 14 covers linear time-series models in detail, including moving average models, autoregressive models, mixed models, and finally models of nonstationary time series. Chapter 15 develops regression methods that can be used to estimate a time-series model, as well as methods of diagnostic checking that can be used to ascertain how well the estimated model "fits" the data. Chapter 16 deals with the computation of the minimum mean-square-error forecast, forecast error, and forecast confidence intervals.

The last chapter of Part Three is devoted entirely to examples of the construction and use of time-series models. After we review the modeling process, we construct models of several economic variables and use them to produce short-term forecasts. Finally, we demonstrate through examples how models can be constructed that combine time-series with regression analysis.

4 USE OF MATHEMATICAL TOOLS

This book is written on a rather elementary level, and can be understood with only a limited knowledge of calculus and no knowledge of matrix algebra. Mathematical derivations and proofs are generally reserved for appendixes or suppressed entirely. In Part One of the book, the development of the regression model in matrix form is included in the appendixes. Thus most if not all of the book should be accessible to advanced undergraduate students as well as graduate students.

It is desirable that the reader of this book have some background in statistics. Although Appendix 2.1 contains a brief review of probability and statistics, the student with *no* background in statistics may find parts of the book somewhat difficult. Typically, this book would be used in an applied econometrics or business-forecasting course which a student would take after completing an introductory course in statistics.

5 ALTERNATIVE USES OF THE BOOK

The book is intended to have a wide spectrum of uses. Curriculum uses include an undergraduate or introductory graduate course in econometrics and an undergraduate or graduate course in business-forecasting. In addition, this book can be of considerable value as a reference book for people doing statistical analyses of economic and business data, or as a text or reference book for the social scientist or business analyst interested in the application of dynamic simulation models to forecasting or policy analysis.

Coverage in an introductory econometrics or business forecasting course must, of course, be dependent to some extent on the background of the student and the goals of the instructor. Emphasis on the use of econometric techniques for purposes of forecasting with econometric models would provide for one focus, but several alternatives are available. We list several alternative uses of the book below, but stress that the great variety of material leaves a good deal of discretion to the instructor planning a course outline.

I UNDERGRADUATE ECONOMETRICS (one semester)

(i) *Standard*

Part One—Chapters 1–6; portions of Chapters 7 and 8 optional
Part Two—Chapter 9

(ii) *Simulation emphasis*

Part One—Chapters 1–6
Part Two—Chapters 10–12

Both courses would omit all matrix appendixes.

II FIRST-YEAR GRADUATE ECONOMETRICS

(i) *One semester*

Part One—Chapters 1–6, Chapters 7 and 8 optional
Part Two—Chapters 9–12

Portions of the above and the appendixes may be optional.

(ii) *Two semesters*
Part One—Chapters 1-8
Part Two—Chapters 9-12
Part Three—Chapters 13-14; portions of Chapters 15-17 optional

Emphasis on either simulation and/or time-series analysis would depend upon the interest of the instructor.

III BUSINESS FORECASTING (graduate or advanced undergraduate)
(i) *One semester*
Part One—Chapter 6 plus review of Chapters 1-5
Part Two—Chapters 10-12
Part Three—Chapters 13-17 (selected portions)

(ii) *Two semesters*
Part One—Chapters 1-6
Part Two—Chapters 9-12
Part Three—Chapters 13-17

IV QUANTITATIVE METHODS FOR POLICY ANALYSIS
(i) *Undergraduate—one semester*
Part One—Chapters 1-6
Part Two—Chapters 10-12

(ii) *Graduate—one semester*
Part One—Chapters 1-6
Part Two—Chapters 9-12

(iii) *Graduate—two semesters*
Part One—Chapters 1-6; Chapters 7 and 8 optional
Part Two—Chapters 9-12
Part Three—Chapters 13-17

The book could also be used for courses in quantitative social science modeling (as taught in departments of sociology or political science). A course in social science modeling that uses this book as a text would probably cover most of Parts One and Two of the book.

6 WHAT DISTINGUISHES THIS BOOK FROM OTHERS?

This book attempts to explain the development and use of quantitative models from a broad perspective. Most textbooks on econometrics develop the single-equation regression model as a self-contained and isolated entity. The reader of such a book often infers that statistical regression models are somehow distinct and independent from other aspects of modeling, such as the analysis of a model's dynamic structure or the use of time-series analysis to forecast one or more exogenous variables in the model. This is certainly not the case, as any practitioner of the art knows. In developing a multi-equation simulation model, for example, one

must be knowledgeable not only of regression methods, but also of how a model's dynamic behavior results from the interaction of its individual equations.

This book develops the techniques and methods for the construction of all three types of models. Thus, the reader becomes aware of the use of single-equation econometrics as a modeling form in itself, as a tool that can be used in the development of multi-equation simulation models, and as a statistical basis for the development of stochastic time-series models for forecasting. The reader also becomes aware that there is more than one type of model, and (we hope) gains an understanding of what models are most preferable for a particular purpose.

We believe that this wide breadth of coverage is desirable. The simulation and time-series techniques that make up Parts Two and Three of this book are usually presented only at an advanced level. We feel that a strength of this book is that the coverage is broad and includes these advanced techniques, but is presented on a level that can be understood and appreciated by the beginning student.

1977

FORECASTING ECONOMIC TIME SERIES

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PREFACE

The academic literature on forecasting is very extensive, but in reality it is not a single body of literature, being rather two virtually nonoverlapping sets concerned with the theoretical aspects of forecasting and the applied aspects. A typical member of one set is very unlikely to mention any member of the other set. It was this realization that motivated the sequence of research projects that eventually resulted in this book. One of the few exceptions to the above statement about nonoverlapping sets is the well-known book by Box and Jenkins, and our own approach owes a lot to their work. However, we have tried to take the state of the art further by introducing new multivariate techniques, by considering questions such as forecast evaluation, and by examining a wider range of forecasting methods, particularly those which have been applied to economic data, on which this present book concentrates. It is one of our aims to further bridge the gap between the theoretical and applied aspects of forecasting.

The analysis of economic data has also been approached from two different philosophies, that proposed by time series analysts and the more classical econometric approach. Although we favor the former, it is quite clear that both approaches have a great deal to contribute and that they need to be brought together to a much greater extent. In the past couple of years, a number of steps have been taken in this direction, and we have

tried to encourage the merger movement in this book by showing how a combined approach to data analysis can lead to potential benefits.

We have many individuals and organizations to warmly thank for their help and encouragement in the preparation of this book and in the research projects that led up to it. Gareth Janacek, John Payne, and Harold Nelson gave considerable help with various aspects of the research. Rick Ashley and Allan Andersen have read and corrected large sections of the manuscript, as have many of our graduate students who had parts of the text inflicted on them for course reading; Alice Newbold prepared all the diagrams for us; Elizabeth Burford and Linda Sykes prepared the final version of the manuscript with great ability and patience; and Robert Young proofread it for us. The Social Science Research Council of the United Kingdom provided the funds to start our research on forecasting in 1970 at the University of Nottingham, and the National Science Foundation of the United States gave us a grant to finally complete the task at the University of California, San Diego. Both universities provided us with excellent facilities, as well as delightful surroundings. Finally, we would like to thank Mike Godfrey, Herman Karreman, and Marc Nerlove for permission to use parts of their own work for illustrations in ours. Of course, we shall have to assume the usual responsibility for those errors that undoubtedly still lurk somewhere in the book.

CONTENTS

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LINEAR LEAST-SQUARES ESTIMATION

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PREFACE

Least-squares estimation is one of those topics that has the happy characteristic of being an apparently inexhaustible source of new problems, new applications, and new results. The reason for this vitality is that despite the seeming narrowness of its title, the subject is intimately connected with the basic structural properties not only of random and deterministic signals, but also of linear and nonlinear systems. This has the dual consequence of frequently bringing new insights from other fields to bear on problems of least-squares estimation and of often showing the usefulness of least-squares ideas in new problems and new applications. We might mention Wiener-Hopf equations, operator factorization, signal detection, martingale theory, matrix algebra, integral equations, scattering theory, and generalized orthogonal polynomial recursions as only some examples of topics that have had both classical and recent mutually fruitful interactions with least-squares theory.

This depth makes several of the classical papers in this field valuable not only for their historical significance but also because they contain insights and ideas that are still capable of further exploration and development. This potential for additional investigation has been one of the criteria used in selecting the papers gathered here. Another criterion has been the relative inaccessibility and sometimes even obscurity of several of the papers, even though this has meant the omission of other works because of their greater availability (some in other volumes in this series). Therefore, I hope that new students as well as those quite learned in this field will find some pleasure and some profit in this collection.

As I have said on numerous occasions, I am grateful to many friends in various countries and in a diversity of fields for shared pleasures in discussing and exploring this rich subject. The broad support of the mathematics department of the Air Force Office of

Scientific Research has been an important ingredient in these efforts. However, for specific assistance with the selections and commentaries in this volume, I am especially indebted to Professor A. M. Yaglom of the Institute of Atmospheric Physics, USSR Academy of Sciences, Moscow.

THOMAS KAILATH

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APPLIED TIME SERIES ANALYSIS

EDITED BY

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PREFACE

The field of time series analysis or signal processing, as some scientists and engineers prefer to call it, is a child of many parents, as this ambiguity of name already suggests. This is a historical and a present fact. A quick search through the current journals of a technical library is likely to turn up time series papers in publications from the fields of acoustics, earthquake engineering, econometrics, electrical engineering, geology, geophysics, guidance and control engineering, mathematics, mechanical engineering, neurophysiology, psychology, and statistics. In such circumstances, it is easy for a time series analyst to be unaware of important developments in the subject occurring in a field remote from his or her own because there are few opportunities for people working in apparently dissimilar fields to get to know one another.

With this situation in mind, the University of Tulsa is sponsoring a series of Applied Time Series Symposia in which distinguished time series researchers from different disciplines give talks about portions of their research they feel might be of general interest. The first such symposium took place in Tulsa, Oklahoma, on May 14–15, 1976. It featured speakers from econometrics, electrical engineering, geophysics, mathematics, and statistics, and attracted 165 participants from these fields and several others.

The speakers were H. Akaike, R. F. Engle, C. W. J. Granger, H. L. Gray, A. G. Houston, R. H. Jones, J. H. Justice, R. T. Lacoss, S. J. Laster, A. V. Oppenheim, E. Parzen, E. A. Robinson, S. Treitel, and G. S. Watson.

The session chairmen were J. B. Bednar, S. E. Elliot, M. R. Foster, and P. M. Robinson.

The present volume contains papers based on talks presented at the symposium. Because of the novelty of their methods, H. L. Gray, A. G. Houston, and F. W. Morgan were invited to prepare a greatly expanded version of their talk for this publication, which they kindly agreed to do. Professor Gray was also kind enough to recommend inclusion of the paper of D. D. McIntire, which illustrates an important facet of these methods. Three papers, those of H. L. Gray, A. G. Houston, and F. W. Morgan, R. H. Jones, and S. J. Laster, include programs. The TIMSAC program packages

of Akaike, Arahata, and Ozaki, discussed in H. Akaike's paper, are available on magnetic tape from the Division of Mathematical Sciences of the University of Tulsa.

The Applied Time Series Symposium was the joint effort of several disciplines and several colleges of the University of Tulsa. Its director was D. F. Findley and its associate director was S. J. Laster, both of whom were given substantial assistance by many colleagues and staff members among whom must be mentioned T. W. Cairns, W. A. Coberly, E. T. Guerrero, N. J. Hyne, D. Murray, and A. R. Soltow. On behalf of all of us, I would like to thank the speakers, session chairmen, and participants for their contributions to a very stimulating conference.

It is my privilege to extend special thanks to the staff of Academic Press for their continued interest in this book, to J. R. Barton for her exceptionally competent work in preparing the manuscript for this book, and to Amoco Research Center and Cities Service Research Laboratory for technical assistance.

David F. Findley

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APPLIED TIME SERIES ANALYSIS

VOLUME I

BASIC TECHNIQUES

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GLOSSARY

a_{1p}	Recursive filter weight, first in cascade implementation
a_{2p}	Recursive filter weight, second in cascade implementation
b	Frequency interval ($=1/NT$)
b_0	Overall nonrecursive scaling factor
b_{0p}	Nonrecursive filter weight, first in cascade implementation
b_{1p}	Nonrecursive filter weight, second in cascade implementation
b_{2p}	Nonrecursive filter weight, third in cascade implementation
B	Signal bandwidth
B_e	Effective resolution bandwidth in PSD calculations
$c(i)$	Convolution function
e	2.7182818...
exp( )	$\exp(x) = e^x$
f	Frequency
F	Folding frequency
$h(i)$	Impulse response function
$H(k)$	Fourier transform of $h(i)$ (the transfer function)
i	Time index; time delay index
Im[ ]	Imaginary part of [ ]
j	$\sqrt{-1}$
k	Frequency index
N	Number of points in the sample
n	Number of degrees of freedom

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DIGITAL FOUNDATIONS OF TIME SERIES ANALYSIS

Vol.1. THE BOX-JENKINS APPROACH

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PREFACE

During the past decade a significant new approach to time series analysis came into importance. The major event which produced this advance was the publication in 1970 of the book Time Series Analysis: Forecasting and Control by George E.P. Box and Gwilym M. Jenkins. The reason why the work of Box and Jenkins is so important is that they open up many new ways to apply time series analysis in order to solve real-world problems. The very vastness of our present technology, relative to the number of scientists and engineers working in the various disciplines, tends to fragment research and development efforts. One of the most severe kinds of fragmentation is that between theory and practice. The work of Box and Jenkins has none of this fragmentation. Instead Box and Jenkins show how the mathematics of time series analysis can be applied to an increasing number of complex problems which require solution in order to meet the needs of our technological society. Few works have succeeded as well as that of Box and Jenkins as a unifying force in bringing everything together in the solution of practical problems.

The present book with the subtitle The Box-Jenkins Approach is Volume 1 of a two-volume work entitled Digital Foundations of Time Series Analysis. This volume gives a self-contained development and explanation of the basic principles of the Box-Jenkins method of time series analysis. By emphasizing the fundamentals instead of advanced techniques, this volume lays the groundwork required for a deeper understanding of the classic book of Box and Jenkins. Unity of treatment is provided by first developing the simple and the multiple regression models of statistics and the linear systems models of engineering. The resulting structure provides the framework for a firm grasp of the Box-Jenkins time series approach.

Advances in integrated circuit technology are having a major impact on the technical areas to which time series analysis can be applied. One of the primary reasons for the importance of digital technology in time series analysis is the ease with which electrical representations of data can be manipulated. The great importance of the digital computer in carrying out time series analysis is reflected in our digital approach throughout this book.

This volume is intended as a text for a one-semester course in time series analysis. The prerequisite is a one-semester course in statistics. An appendix on matrix algebra is given for those readers who have not had an introduction to this subject. Upon completion of a course based on this volume, the student would be prepared for a course in time series analysis at the level of the book by Box and Jenkins. He would also be prepared to start applying time series methods to the solution of real world problems. In this respect this book is also useful as a reference for those engaged in empirical statistical work which involves the analysis of time series data.

Volume 2 of this work has the subtitle Wave-Equation Space-Time Processing. Volume 2 brings together analytic and digital methods to process data associated with multidimensional models, particularly those requiring both space and time variables for their descriptions. The approach is to consolidate the space-time representation of physical processes involving wave motion by means of the wave equation. Typical problem areas are those associated with sonar, where there is a need for simultaneously fulfilling a number of requirements such as the determination of multiple signal sources, their location in space, and their identification. Similar problems arise in radar, optics, radio astronomy, geophysics, and medicine.

PREFACE

Another important area of time series analysis is that associated with
ntrol theory in the areas of state-space modeling and Kalman filtering. The
o volumes of this work do not cover this field, and instead we refer the
ader to the book Discrete Techniques of Parameter Estimation by Jerry Mendel
ublished by Marcel Dekkes, New York, 1973).

We are indebted to many people in the writing of this work. Our sincere
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deconvolution of geophysical time series in the exploration for oil and natural gas

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ANALYSIS OF ECONOMIC TIME SERIES

A Synthesis

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TIME SERIES ANALYSIS: Regression Techniques

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Editor's Introduction

Social scientists have become increasingly concerned with the analysis of change. Such an emphasis is the focus of TIME SERIES ANALYSIS, which examines techniques based on regression analysis for the study of change.* Many techniques are employed in the analysis of change, but regression analysis is the one most commonly employed and thus in the greatest need of a clear explication.

Research employing statistical methods will always have at least one of two fundamental types of data set: (1) *cross-sectional* data, in which the researcher has observations on a set of variables at a given point in time across many nations, states, counties, cities, or other units of analysis; or (2) *time-series* data, in which one has a set of observations on some variables for the same unit of analysis (such as a nation, state, etc.) over a series of time points (days, months, years, etc.). For either type of data, one can employ techniques based on regression analysis. Why, then, a special paper on regression techniques in TIME SERIES ANALYSIS?

The answer lies in the nature of time-series data. The regression model is based on a specific set of assumptions, relatively few in nature but very important if one is not to make erroneous inferences from the analysis of a data set. When we speak in terms of change over time, we may think of the analogy of "time's arrow," i.e., events in time move in one direction—forward. Social science data also move in one direction when examined over time in many cases. The world's population has been continually increasing over time; so has public spending by various units of government. When there is a general pattern of increases in the value of variables over time, however, some problems with regression estimation arise. In particular, it is likely that the assumption of independent error terms for succeeding observations may be

*See Herbert B. Asher (1976) *Causal Modeling*, in this series, for a discussion of the regression techniques and the assumptions behind the model which Ostrom also presents in this paper.

violated. This is considerably less likely to occur in cross-sectional data analysis, where the order in which the observations are used to derive the regression estimates is rarely a matter of concern. Thus, one critical difference between cross-sectional analysis and TIME SERIES ANALYSIS is that, for the latter, it is critical that the data be processed in the order of the time periods involved. For yearly data, we enter 1952, 1953, 1954, 1955, 1956, etc. rather than 1955, 1952, 1954, 1956, etc. The way we process the data is important for two reasons: (1) the statistical properties of the regression estimators may be affected by the order in which we enter the cases, particularly if many of the variables are consistently increasing over time (or "tracking" like a missile, always in an upward direction);** and (2) as the author notes, TIME SERIES ANALYSIS allows us not only to get estimates of a regression equation, but also to make *forecasts* into the future from data in the past. If we were to order our data randomly, we might as well assume that the future can "come before" the past!

Charles W. Ostrom, Jr. demonstrates how regression techniques can be employed for both hypothesis testing and forecasting in TIME SERIES ANALYSIS. His major example concerns defense expenditures in the United States and the Soviet Union. This is a question that has interested students of politics, economics, and even applied mathematics throughout the twentieth century, with the earliest sophisticated statistical models developed by the applied mathematician Lewis Richardson.*** The subject of TIME SERIES ANALYSIS is important to all social scientists, however.

- Political scientists, in addition to studying changing patterns of expenditures on armaments and domestic social welfare programs (among other categories), have been concerned with changing patterns of party strength in presidential and congressional elections.
- Historians have studied *electoral results over time*, and examined changing patterns of migration and socioeconomic mobility in many societies, including the U.S. These studies often yield findings about social structures that have gone against the conventional wisdom of other scholars who relied exclusively on the writings of observers of previous eras rather than examining the available data on social structures. The growth of quantitative methods in history, together with a changing social consciousness, has led many historians to seek out data sources previously thought unavailable, including census records of states and municipalities on the number and working conditions of slaves and freed slaves in nineteenth-century America.****
- Economists have been among the pioneers in developing TIME SERIES ANALYSIS (generally considered to be a branch of "econometrics," statistical methodology developed by economists). They have developed large-scale models of the economies of the U.S. and many other nations, with particular emphasis on changes over time and with forecasts of future trends.
- Sociologists have been concerned with many of the same demographic analyses from census tracts as historians. They also use TIME SERIES ANALYSIS to examine changes in life styles over generations, including questions of the differing roles for organized religion, sexual and racial characteristics, and values in types of behavior (such as family relations and individual voting choices).*****
- Psychologists and educators have examined changes in achievement levels of students. Are such changes related to the age of the student, changing racial and/or sexual roles, changes in curriculum or in the larger environment of the society (such as the amount of time spent watching television)?

What, then, is the scope of TIME SERIES ANALYSIS? Anything that moves! More appropriately, the answer should be anything subject to change. Thus, TIME SERIES ANALYSIS is fundamental to most of the questions posed in social science research. History is, after all, the study of change; the other social sciences merely examine what might be called "contemporary history." As Ostrom notes, the techniques discussed here provide an introduction to TIME SERIES ANALYSIS. Even within the regression framework, there are additional topics interested students should examine. Ostrom provides brief descriptions of these other techniques in a field where developments in statistical theory are constantly occurring. Thus the methodology, like the subjects of the analysis, is itself always changing.

—E. M. Uslaner, Series Editor

*This terminology is taken from Ronald J. Wonnacott and Thomas H. Wonnacott (1970) *Econometrics*. New York: John Wiley.

**While Richardson did his work over a half century ago, the references in the international relations literature are all to two 1960 reprints: (1960a) *Arms and Insecurity* and (1960b) *Statistics of Deadly Quarrels*, both published in Chicago by Quadrangle Books.

****A highly controversial work in this area, from both the perspective of the methods employed and the conclusions reached, is Robert W. Fogel and Stanley F. Engerman (1974), *Time on the Cross*. Boston: Little, Brown, and Co., in two volumes.

*****On generational change and how to distinguish it from other types of change, see Norval D. Glenn (1977) *Cohort Analysis* in this series.

TIME SERIES ANALYSIS:
Regression Techniques

CHARLES W. OSTROM, JR.
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1. INTRODUCTION

This monograph is designed to serve as an in-depth introduction to a variation of the basic regression model that utilizes data of a special type—time series. A collection of data $X_t (t = 1, 2, \ldots, T)$ with the interval between X_t and X_{t+1} being fixed and constant is referred to as a time series. In short, the order of the observations is of extreme importance—we are interested not only in the particular values of the observations but also in the order in which they appear. For example, series relating to U.S. defense expenditures, amount of war in the international system, presidential support, and unemployment meet the requirements of time series analysis. Thus, whereas in most treatments of regression (e.g., Wonnacott and Wonnacott, 1970; Kmenta, 1971; Johnston, 1972; Kelejian and Oates, 1974; Uslaner, 1978) the ordering of the observations has been irrelevant, in this paper it is of prime importance.

Given data that are in a specified temporal ordering, it is possible to raise questions concerning how the variable has behaved in the past and how it is likely to behave in the future. The great advantage of time series regression analysis is that it is possible to both explain the past and predict the future behavior of variables of interest. Thus the past history of a time series is called upon to do double duty (Nelson, 1973:19): "first, it must inform us about the particular mechanism which describes the evolution through time and second, it allows us to put that mechanism to use in forecasting the future." As can be seen, both of these efforts are predicated upon being able to correctly postulate a model and estimate its parameters. For example, the decision by the U.S. with respect to how much to spend for defense is of great concern to the President, Congress, the public, and other world leaders. As a result it is imperative that we try to understand how such a decision is made and what the future ramifications of the decision mechanism are likely to be. In an effort to provide some continuity to the technical discussion, we shall return to this example throughout this paper.

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